Intuitionistic Truth-knowledge Symmetric Bilattices for Uncertainty in Intelligent Systems

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Abstract—Differently from pure probability theory the common uncertain information is perception-based and imprecise [1]. Human belief, confidence level, etc... are approximate human perceptions and the intelligent systems need a general approximate reasoning logic for them. We propose a family of intuitionistic bilattices with full truth-knowledge duality to be used in logic programming for such uncertain information. The simplest of them, based on intuitionistic truth-functionally complete extension of Belnap’s 4-valued bilattice, can be used in paraconsistent programming, that is, for knowledge bases with incomplete and inconsistent information. The other two families are useful for an approximate logic theory where the uncertainty in the knowledge about a piece of information is in the form of human granulation cognition types: as an interval-probability belief or as a confidence level. Such logic programs can be parameterized by different kinds of probabilistic conjunctive/disjunctive strategies for their rules, based on intuitionistic implication, which express the user perception-based correlation between observed knowledge facts.

I. Introduction

In mathematics and logic, an intuitionistic type theory, or constructive type theory, is, most broadly, any type theory done in accordance with the principles of mathematical constructivism. There is particular interest in pure intuitionistic type theories that can serve as a foundation of mathematics. Many constructive type theorists, especially those working in foundations, want to develop a type theory that can serve at the same time as a mathematical language and a logic programming language. The Curry Howard isomorphism suggests an identification of propositions and types in intuitionistic logic, so that type theory encompasses the (intuitionistic) predicate calculus, while also providing an alternative to set theory.

There is a close relation between Logic Programming and inductive definitions which are forms of constructive knowledge [2], [3]. Constructive information defines a collection of facts through a constructive process of iterating a recursive recipe. This recipe defines new instances of this collection in terms of presence (and sometimes the absence) of other facts of the collection. In the context of mathematics, constructive information appears by excellence in inductive definitions. Not a coincidence, inductive definitions have been studied in constructive mathematics and intuitionistic logic, in particular in the sub-areas of Inductive and Definition logics, Iterated Inductive Definition logics and Fixpoint logics.

In the past few decades, many classical as well as non-classical techniques for modelling the reasoning of intelligent systems have been proposed. Several frameworks, based on the many-valued logic with truth/knowledge preorders (lattices), for manipulating uncertainty, incompleteness, and inconsistency have been proposed in the form of extensions to classical logic programming and deductive databases. Such truth/knowledge ordered lattices can be integrated into the unique structure called bilattice [4], [5], [6], and Ginsberg has shown that the same theorem prover can be used to simulate reasoning in first order logic, default logic, prioritized default logic and assumption truth maintenance system.

Bilattices are algebras with two separate lattice structures. They have been used for a denotational semantics for systems of inference that arise in artificial intelligence and knowledge-based Logic Programming. In particular, they have been used to provide a general framework for an efficient procedural semantics of logic programming languages that can deal with incomplete as well as contradictory information. Originally, Ginsberg [6] suggested using bilattices as the underlying framework for various AI inference systems including those based on default logics, truth maintenance systems, probabilistic logics, and others. These ideas were later pursued in the context of Logic Programming semantics [5], [7]. A variant of Fitting’s extension of logic programming to bilattices was used to deal with a form of negation as failure as well as a second explicit negation in logic programs [8]. In [9] bilattices were extended to include a third ordering (called precision ordering) in order to deal with varying degrees of belief and doubt in probabilistic deductive databases. In this paper we will consider different belief truth structures (types) for knowledge systems: the simplest 4-valued bilattice, belief probability intervals [10], [11], [12] with the lower and upper probability boundary and also confidence-level quantified data [9] where belief and doubt are considered separately for each fact. All such logic systems are many-valued and based on a bilattices (lattices with truth and knowledge ordering), and such kind of applications are the main motivation for this work: the kind of intuitionistic bilattices introduced in this paper are dedicated for development of logic programs which deal with such more complex truth values for uncertain information.

Studies of the algebraic structure of bilattices have led to practical results, particularly in reducing the computational complexity of bilattice based Logic Programs. It is therefore important to further study bilattices and their algebraic properties. This, in part, is the motivation behind the present work. As first contribution we will extend these research by introduction of a new modal operator for bilattices which as we will show plays the fundamental rule in the truth/knowledge bilattice’s duality: in logic programs and their least-fixpoint semantics.

As second contribution here one particular family of intuitionistic bilattices that has nice mathematical properties is
considered [13]. Semantically, the basic notion is that of relative pseudo complement from the algebraic semantics of intuitionistic logic [14]. Such implications may be considered as candidates for nested implications in bodies of rules (see, for example, in [15]).

The last contribution is given by a family of parameterized bilattice operators for different probabilistic strategies in a belief and confidence-level based logic programming.

The plan of this paper is the following: after a brief introduction to bilattices, in Section 3 we introduce the intuitionistic implication for bilattices with respective Modus Ponens and Deduction theorem properties for both truth and knowledge orderings. In Section 3 we formalize the strict subset of dual bilattices (D-bilattices) and we present the family of intuitionistic D-bilattices with full truth-knowledge duality: the simplest 4-valued bilattice, world-based bilattices and higher-order bilattices, and is demonstrated the full duality for least fixpoint semantics of logic programs based on D-bilattices. Finally, in Section 4 we discuss the application of these intuitionistic D-bilattices in Logic Programming: the two such D-bilattices are formally introduced, the belief and the confidence-level bilattices, which are also parameterized for a number of possible probabilistic conjunctive strategies.

A. Introduction to Bilattices

Bilattice theory is a ramification of multi-valued logic by considering both truth \( \leq \) and knowledge \( \leq k \) partial orderings. Given two truth values \( x \) and \( y \), if \( x \leq y \) then \( y \) is at least as true as \( x \). The two operations corresponding to this ordering (t-lattice) are the meet (greatest lower bound) \( \wedge \) and the join (least upper bound) \( \vee \).

For the knowledge (or precision in probabilistic theory) ordering \( x \leq k y \) means that \( y \) is more precise than \( x \). The two operations \( \otimes \) and \( \oplus \) correspond to the greatest lower bound and least upper bound respectively in the knowledge 

Definition 1: (Ginsberg [6]) A bilattice \( B \) is defined as a sextuple \( (B, \wedge, \vee, \odot, \oplus, \neg) \), such that:
1. The t-lattice \( (B, \wedge, \vee) \) and the k-lattice \( (B, \otimes, \oplus) \) are both complete lattices.
2. \( \neg : B \rightarrow B \) is an involution (\( \neg \neg \) is the identity) mapping such that: \( \neg \) is lattice homomorphism from \( (B, \wedge, \vee) \) to \( (B, \wedge, \vee) \) and \( (B, \otimes, \oplus) \) to itself. \( \Box \)

We can see such (bi)lattice structures as algebras, denoted \( (B, \alpha) \), with a carrier set \( B \) of truth values and the set of \( \alpha \) of n-ary operations,

\[ \alpha_i : \prod B \rightarrow B, \alpha_k : \prod B \rightarrow B \text{ , where } \alpha_i = \{\wedge, \vee\}, \]

\[ \alpha_k = \{\otimes, \oplus\} \text{ and } \prod B = B \times B \text{ cartesian product.} \]

Any algebra homomorphism \( h : (B, \alpha) \rightarrow (B, \beta) \) is a mapping such that for any term \( f(a_1, \ldots, a_n) \), where \( f \in \alpha_i \) is a n-ary operation of the first algebra and \( a_i \in B \), \( h(f(a_1, \ldots, a_n)) = h(f)(h(a_1), \ldots, h(a_n)) \), where \( h(f) \in \alpha_k \) is the correspondent n-ary operation of the second algebra.

For example for \( h = \neg : (B, \alpha) \rightarrow (B, \beta) \), where \( \beta = \{\vee, \wedge\} \), we have that \( \neg(\wedge(a, b)) \) (that is, \( \neg(a \wedge b) \)) is equal to \( \neg(\neg(a)) \neg(b) = \neg a \vee \neg b \), where \( \neg(\wedge) = \vee \), that is, this homomorphism corresponds to De Morgan law. It is easy to verify that this homomorphism \( \neg \) inverts the truth partial ordering, that is, if \( x \leq y \) then \( \neg x \geq \neg y \), while it preserves the knowledge preordering.

In any bilattice with the top and bottom truth values \( t \) and \( f \) respectively, we have that \( \neg t = f \), \( \neg f = t \), while in any nontrivial bilattice they are k-incomparable.

The smallest bilattice is the trivial bilattice, in which \( B \) consists of a single value, so that it is useless.

The smallest nontrivial bilattice is Belnap’s 4-valued bilattice \( [4] \) \( B_4 = \{ t, f, \bot, \top \} \) where \( t \) is true, \( f \) is false, \( \top \) is inconsistent (both true and false) or possible, and \( \bot \) is unknown. As Belnap observed, these values can be given two natural orders: truth order, \( \leq \), and knowledge order, \( \leq k \), such that \( f \leq t \leq \bot \), \( f \leq \bot \leq t \) and \( \bot \leq k f \leq k t \), \( \bot \leq k t \leq k \top \). This two orderings define corresponding equivalences \( =_t \) (a = b if \( a \leq t \) and \( b \leq t \)) and \( =_k \) (a = b if \( a \leq k b \) and \( b \leq k a \)). Thus any two members \( \alpha, \beta \) in a bilattice are equal, \( \alpha = \beta \), if and only if (shortly ‘iff’) \( \alpha =_t \beta \) and \( \alpha =_k \beta \). Meet and join operators under \( \leq \) are denoted \( \wedge \) and \( \vee \); they are natural generalizations of the usual conjunction and disjunction notions. Meet and join under \( \leq k \) are denoted \( \odot \) (consensus, because it produces the most information that two truth values can agree on) and \( \oplus \) (gullibility, it accepts anything it’s told), such that hold: \( f \odot k t = f \), \( f \oplus k t = \top \), \( \top \wedge k \bot = f \) and \( \top \vee k \bot = t \).

There is a natural notion of truth negation, denoted \( \neg \) (reverses the \( \leq \) ordering, while preserving the \( \leq k \) ordering): switching \( f \) and \( t \), leaving \( \bot \) and \( \top \), and corresponding knowledge negation (conflation) [5], denoted \( \neg k \), (reverses the \( \leq k \) ordering, while preserving the \( \leq \) ordering), switching \( \bot \) and \( \top \), leaving \( f \) and \( t \). These two kind of negation commute: \( \neg \neg x = \neg x \) for every member \( x \) of a bilattice.

It turns out that the operations \( \wedge, \vee \) and \( \neg \), restricted to \( \{ f, t, \bot, \top \} \) are exactly those of Kleene’s strong 3-valued logic. A more general information about this smallest bilattice may be found in [5]. The Belnap’s 4-valued bilattice is infinitary distributive.

II. INTUITIONISTIC BILATTICES: MODUS PONENS AND DEDUCTION THEOREM

In mathematics and logic, an intuitionistic type theory, or constructive type theory, is, most broadly, any type theory done in accordance with the principles of mathematical constructivism. Proof-theoretic semantics is an approach to the semantics of logic that attempts to locate the meaning of propositions and logical connectives not in terms of interpretations, as in Tarskian approaches to semantics, but in the role that the proposition or logical connective plays within the system of inference. Gerhard Gentzen is the founder of proof-theoretic semantics, providing the formal basis for it in his account of cut-elimination for the sequent calculus, and some provocative philosophical remarks about locating the meaning of logical connectives in their introduction rules within natural deduction. Dag Prawitz extended Gentzen’s notion of analytic proof to natural deduction, and suggested that the value of a proof in natural deduction may be understood as its
normal form. This idea lies at the basis of the Curry-Howard isomorphism, and this idea of intuitionistic type theory. His inversion principle lies at the heart of most modern accounts of proof-theoretic semantics.

That is the reason why we introduced intuitionistic implication in the bilattice theory: the nature of logic programming is constructive. For any ground rule $A ← B_1, ..., B_n$ of a many-valued logic program $P$, we derive the truth value $v(A)$ of the head of its rule from the truth value $v(B_1 ∧ \ldots ∧ B_n)$ of its body, during the calculation of the least fixed point model of such logic program. $(v$ is a valuation, that is, an extension of a Herbrand interpretation to all ground formulae). More over we require that such ground rule is satisfied for a given valuation $v$ iff $v(A) ≥_t v(B_1 ∧ \ldots ∧ B_n)$, that is iff $A ← B_1, ..., B_n$ is true, and that is just the case of the intuitionistic implication: 

$$\alpha ⇀ \beta = \max_t \{ \delta | \delta ∧ \alpha ≤_t \beta \} = t$$

that is $t ≤_t \alpha ⇀ \beta$ iff $\alpha ≤_t \beta$, which corresponds to the many-valued Deduction theorem (when we replace the $\alpha ⇀ \beta$ modal relation by the entailment $\vdash$).

For the many-valued Modus ponens we have $(\alpha ⇀ \beta) ∧ \alpha = \max_t \{ \delta | \delta ∧ \alpha ≤_t \beta \}$, that is $(\alpha ⇀ \beta) ∧ \alpha \vdash \beta$. The obtained D-bilattice for $\mathcal{L}$, can be directly generalized into the category theory. Let $\mathcal{L}$ be the language obtained from a given first-order vocabulary that contains a finite nonempty set of predicate symbols and finite nonempty set of constants (the given first-order vocabulary that contains a finite nonempty set $\{\land, \lor, \neg, \top, \bot\}$). We are able to define two "truth-knowledge-dual" categories with the set of objects equal to the set of formulæ in $\mathcal{L}$, and with arrows as follows:

1. The poset category $\mathcal{L}_T$, where arrows are inverse of truth partial ordering, that is, if $\varphi ≤_t \psi$, with $\varphi, \psi ∈ \mathcal{L}$, then there is an arrow from $\varphi$ to $\psi$.
2. The poset category $\mathcal{L}_K$, where arrows are inverse of knowledge partial ordering, that is, if $\varphi ≤_k \psi$, with $\varphi, \psi ∈ \mathcal{L}$, then there is an arrow from $\varphi$ to $\psi$.

It is easy to verify that such categories are equivalent, and Cartesian Closed Categories: The Cartesian product corresponds to the joint operation in the truth and knowledge lattices, $\land$ and $\oplus$, respectively, with projections, $\varphi \land \psi \leq_t \varphi \land \psi$, and exponential objects, such that, for example, in $\mathcal{L}_T$ and $\mathcal{L}_K$, the following diagrams commute

That is, if $\varphi ∧ \psi ≤_t \phi$, then $\varphi ≤_t \psi → \phi = \max_t \{ \varphi ∧ \psi ≤_t \phi \}$ (Deduction theorem), while the upper arrow $(\psi → \phi) ∧ \psi ≤_t \phi$ is a generalized many-valued Modus Ponens, which, when $\psi → \phi$ are true reduces to the ordinary Modus Ponens (that is, also $\phi$ must be true).

The developed intuitionistic bilattices can be usefully applied to paraconsistent knowledge bases also: classical logic predicts that everything (thus nothing useful at all) follows from inconsistency, while a paraconsistent logic tries to avoid such an explosion: intuitionistic implication blocks the following inference schemas, $\alpha ∧ ¬\alpha → \beta$, $(\alpha → \beta) ∧ \alpha ∧ ¬\beta → \delta$ and $(\alpha → \beta) ∧ ¬¬(\alpha → \beta) → \delta$.

It is easy to verify that the intuitionistic implication blocks such inference schemas.

III. BILATTICES AND TRUTH/KNOWLEDGE SYMMETRY

In this section we will introduce a family of D-bilattices with intuitionistic implication and with a perfect duality between its truth and knowledge lattices, denominated t-lattice and k-lattice respectively.

The intuitive meaning for such D-bilattices is that any property defined over the truth ordering (t-lattice) can be equivalently defined over the knowledge ordering (k-lattice), and vice versa.

That is, they are perfect dual images.

Definition 2: A D-bilattice $B$ is composed by two residuated lattices extended by a modal operator: a t-lattice $(B, ≤_t, α_t)$, with $α_t = \{\land, \lor, ¬\}$, and a k-lattice $(B, ≤_k, α_k)$, with $α_k = \{\land, \lor, ¬\}$, such that:

1. The t-lattice $(B, ≤_t, α_t)$ and the k-lattice $(B, ≤_k, α_k)$ are both complete lattices.
2. $\land$ and $\land$ are meet operators w.r.t. the ordering of these two lattices. We denote the join operators by $∨$ and $∨$ respectively.
3. $¬$ and $¬$ are relative pseudo-complements of these two lattices, with negation operators $¬_t$ and $¬_k$ (pseudo-complements).
4. $¬$ and $¬$ are involutions modal operators (i.e., $\sim$ and $\sim$ are identities), conjugate to itself ($\sim_1 \sim_1$ and $\sim \sim$), such that:

4.1. are normal, that is, for the greatest element $α_t$ and $α_k$ of these two lattices, $¬(α_t) = α_t$, and $¬(α_k) = α_k$;
4.2. are multiplicative, that is, $(x ∧ y) = ¬(¬x ∨ ¬y)$ and $¬(x ∨ y) = ¬x ∨ ¬y$.
4.3 satisfy De Morgan law between join and meet operators, $x ∨ y = ¬(¬x ∧ ¬y)$ and $x ∨ y = ¬(¬x ∨ ¬y)$.

5. There exists the duality modal operator $\partial$ over $B$ which is an isomorphism between these two lattices $\partial : (B, ≤_t, α_t) \cong (B, ≤_k, α_k)$ also, which preserves the ordering, that is, if $x ≤_t y$ then $\partial(x) ≤_k \partial(y)$.

Informally, these dual lattices are monadic extensions of Heyting algebras.

Notice that important point of the definition above is the point 4 which introduces the two modal operators $¬_t$ and $¬_k$. These conjugate modal operators are belief operators which model the S5 autoepistemic modal logic. They are very important in the case when the intuitionistic negation operators $¬_t, ¬_k$ are non monotonic for both orderings and can not be used in order to define the fixpoint semantics for logic programs (as,
for example in Belnap’s bilattice): in that case, for example
\( \neg A \) is not monotonic w.r.t. the truth ordering, but is
monotonic w.r.t. the knowledge ordering, and normal 3-valued
logic programs can use it as default (or epistemic) negation
with fixpoint semantics computed w.r.t. knowledge ordering.
For example, consider the clause \( A \leftarrow B_1, \neg B_2 \), where \( \neg \)
is the epistemic (default) negation in logic programs, such
that \( \neg = \neg_{\text{epistemic}} \), where \( \sim \) is a modal belief operator and
\( \neg_t \) ‘classical’ (i.e., intuitionistic) negation operator in \( \alpha_t \) (i.e.,
standard way to represent normal (3-valued) logic programs
as autoepistemic logic programs).

The point 5 instead formally introduces the duality operator \( \neg \)
for the truth/knowledge orderings in the D-bilattices.
We will show that such family of D-bilattices is very sig-
ificant for application in logic, and that also the simplest
non trivial bilattice is a member of this family. The fact that
such isomorphism preserves the ordering is very important,
so that if we define some fixpoint semantics for logic programs
w.r.t. the some truth-ordering monotonic operator, we are able
to define such semantics w.r.t. the correspondent knowledge-
ordering monotonic operator, and viceversa.

A. The smallest nontrivial D-bilattice
The smallest non trivial bilattice is a 4-valued Belnap’s
bilattice, particularly important for the knowledge systems
with inconsistent and incomplete information. It can be seen
as the minimal many-valued logic system where we are
able to deal with incomplete and inconsistent information:
in [16] is presented an autoepistemic logic programming,
with the Moore’s modal operator, where an intuitive and
natural approach is used to resolve inconsistency, with the
simple monotonic (w.r.t. the knowledge ordering) ”immediate
consequence operator” for the least-fixpoint computation of
their Herbrand models.

The following propositions show that Belnap’s bilattice has
the perfect duality properties and that the duality extension of
the original Belnap’s bilattice is truth-functionally complete.

**Proposition 1:** Belnap’s bilattice \( B_4 \) is a D-bilattice. □

**Proof:** It is enough to prove for the points 3 and 5. We define
the modal dual operation \( \neg \) as follows:
\[ \neg t = \tau, \quad \neg \tau = t, \quad \neg f = \bot, \quad \text{and} \quad \neg \bot = f. \]

It is easy to verify that it is an involution, \( \neg \neg = \text{id} \), and that
hold \( \neg(x \land y) = \neg x \lor \neg y \), \( \neg(x \lor y) = \neg x \land \neg y \), and \( \neg \sim x = \neg \sim x \), and \( \neg(x \lor y) = \neg x \lor \neg y \), \( \neg(x \land y) = \neg x \land \neg y \), and \( \neg \sim x = \neg \sim x \).

Just because of this isomorphism \( \neg \), the t-lattice (or equivalently,
\( \neg \sim \)) completely defines the D-bilattice.

We can see that the conflation (knowledge negation) operator \( \sim \)
is the epistemic belief operator “it is believed that” for a
bilattice, which extends the 2-valued belief of the autoepistemic
logic as follows:

- if \( A \) is true than "it is believed that \( A \)”, i.e., \( \sim A \), is true.
- if \( A \) is false than "it is believed that \( A \)”, i.e., \( \sim A \), is false.
- if \( A \) is unknown than "it is believed that \( A \)”, i.e., \( \sim A \), is inconsistent: it is really inconsistent to believe in
something that in unknown.

- if \( A \) is inconsistent (that is both true and false) than "it is
believed that \( A \)”, i.e., \( \sim A \), is unknown: really, we

Notice that in this 4-valued framework by assigning to some
inconsistent fact the value \( \tau \) we avoid the explosive inconsis-
tency; that is the good feature of Belnap’s based autoepistemic
logic, differently for the classic 2-valued logic, and that is the
reason that we are able to use it in the cases where mutually-
inconsistent information can happen, as, for example in data
integration of different data sources [17], [16]. So, as in 2-
valued case, the epistemic negation is the (intuitionistic) nega-
tion of the belief, that is \( \sim = \neg_t \sim_t \). Dually, for the knowledge
ordered lattice, holds that the (epistemic) knowledge negation
is the negation of the knowledge-belief, that is \( \sim = \neg_k \sim_k \).

It is easy to verify that, differently from classic implication,
the \( x \rightarrow y \) is different from \( \neg x \lor y \), and the more strong
requirement holds:

**Proposition 2:** The implication \( \rightarrow \) and the duality operator \( \neg \)
on the bilattice \( B_4 \) cannot be written in terms of the existing
bilattice functions \( \land, \lor, \neg, \odot \) and \( \neg \) defined on \( B_4 \). □

**Proof:** The intuitionistic negation \( \neg \) is defined by \( \neg x = x \rightarrow f \). Let us prove that it cannot be written in terms of
\( \land, \lor, \neg, \odot \) and \( \neg \). We have that all functions in \( \land, \lor, \neg, \odot \)
and \( \neg \) distribute with respect to the operation \( \odot \), so that if
\( \kappa \) is any modal operator constructed from them we will have
\( \kappa(\bot) = \kappa(f \odot t) = \kappa(f) \odot \kappa(t) \) (see Proposition 4.1 in [18]).
However, \( \neg t \bot = \tau \), while \( \neg f \odot \neg t = t \odot f = \bot \neq \tau \).
Also, \( \neg \bot = f \), while \( \neg f \odot \neg \bot = \bot \odot f \neq \bot \).

We are able to define other modal operators over D-bilattices,
as, for example, Moore’s [19] modal operator \( M \), where \( M x \)
is intended to capture the notion of “I know that \( x \)”. In the
4-valued framework this is related to the following unary
mapping on the bilattice \( B_4 \) (in [18]):
\[ M(x) = t, \quad \text{if} \quad x \in \{ t, \tau \}; \quad f, \quad \text{otherwise.} \]

**Proposition 3:** In Belnap’s bilattice Moore’s modal operator is defined by
\[ M(x) = \partial(\neg(x \rightarrow x)) \neg \partial(\neg(x \rightarrow x)). \]

Now we will prove the truth-functionally completeness.

**Proposition 4:** Every modal operator on the bilattice \( B_4 \) can
be written as a combination of the operators \( \land, \neg, \odot \) and
the constant function \( \bot \). □

**Proof:** For the truth-functionality completeness we have that:
the Moore’s modal operator is defined by \( M(x) = \partial(\neg(x \neg \bot \neg \rightarrow x) \neg \partial(\neg(x \neg \bot \rightarrow x)) \), while the consensus
is defined by \( x \odot y = \partial(\neg(x \neg y) \neg y) \), thus, from Proposition 4.2 [18] this proof is concluded. □

The upshot of this proposition is that no additional operators
are needed, that is, the set of operators in this D-bilattice is
truth-complete. That is, for any logic program ( also with
Moore’s or other modal operators) it is enough to use the
conjunction, epistemic negation, intuitionistic implication and
duality operator.

B. Duality in world-based bilattices

Ginsberg [6] defined a world-based bilattices, considering
a collection of worlds \( W \), where by world we mean some
possible way of how things might be:
Definition 3: [6] A pair \([U, V] \in \mathcal{P}(W) \times \mathcal{P}(W)\) of subsets of \(W\) (here \(\mathcal{P}(W)\) denotes the powerset of the set \(W\)) express truth of some sentence \(p\), with \(\leq_t, \leq_k\) truth and knowledge preorders relatively, as follows:

1. \(U\) is a set of worlds where \(p\) is true, \(V\) is a set of worlds where \(p\) is false, \(P = U \cap V\) is a set where \(p\) is inconsistent (both true and false), and \(W = (U \cup V)\) where \(p\) is unknown.
2. \([U, V] \leq_t \{U_1, V_1\}\) if \(U \subseteq U_1\) and \(V \subseteq V_1\).
3. \([U, V] \leq_k \{U_1, V_1\}\) if \(U \subseteq U_1\) and \(V \subseteq V_1\).

The bilattice operations associated with \(\leq_t\) and \(\leq_k\) are:

4. \([U, V] \cap [U_1, V_1] = [U \cap U_1, V \cap V_1]\).
5. \([U, V] \cup [U_1, V_1] = [U \cup U_1, V \cup V_1]\).
6. \([U, V] \oplus [U_1, V_1] = [U \cup U_1, V \cup V_1]\).
7. \(-[U, V] = [V, U]\).

Let denote by \(B_W\) the set \(\mathcal{P}(W) \times \mathcal{P}(W)\), then \((B_W, \wedge, \vee, \oplus, \ominus, \neg)\) is a bilattice.

If we take that \(W\) is equal to Herbrand base \(H_P\) of a logic program \(P\), then a member \([U, V] \in B_W\), where \(U\) is the subset of true ground atoms and \(V\) the subset of false ground atoms w.r.t some Herbrand interpretation \(I = v_B : H_P \rightarrow B\), then this member is set-based representation of this Herbrand interpretation, and \(B_W\) is isomorphic (that is, equivalent) to the functional space \(B^{H_P}\). Let us show that this functional space \(B^{H_P}\), that is, the world-based bilattice \(B_W\) is a D-bilattice.

Proposition 5: The world-based bilattice is a D-bilattice composed by two lattices, a \(t\)-lattice \((B_W, \alpha_t)\), with \(\alpha_t = \{\wedge, \neg, \rightarrow\}\), and a \(k\)-lattice \((B_W, \alpha_k)\), with \(\alpha_k = \{\ominus, \ominus, \neg\}\), such that:

1. Intuitionistic implication:
   \([U, V] \rightarrow [U_1, V_1] = [U \cup U_1, V \cap V_1]\).
2. Duality operation: \(\partial[U, V] = [\overline{V}, \overline{U}]\).
3. Knowledge negation (modal belief operator):
   \(-[U, V] = ([\overline{U}, \overline{V}] ,\overline{S})\), where \(S = W - S\) is the set complement of \(S\).

Proof: For intuitionistic implication holds that:
\[\{U, V\} \rightarrow \{U_1, V_1\} = \max\\{\{U_2, V_2\} | \{U_2, V_2\} \cap [U_1, V_1]\} = \max\\{\{U_2, V_2\} | \{U_2, V_2\} \subseteq U_1\} \land \{V_2 | V_2 \subseteq V_1\} = [U \cup U_1, V \cap V_1]\.

Now we will show that if the basic bilattice \(B\) is a D-bilattice, then also the higher level bilattice (Functional space of Herbrand interpretations) is a D-bilattice.

Lemma 1: The Function space bilattice is a D-bilattice composed by two lattices, a \(t\)-lattice \((B^{H_P}, \alpha_t)\), with \(\alpha_t = \{\wedge, \neg, \rightarrow\}\), and a \(k\)-lattice \((B^{H_P}, \alpha_k)\), with \(\alpha_k = \{\ominus, \ominus, \neg\}\), such that:

1. Implication: for truth
   \[v_B \rightarrow w_B = \max_t\{u_B | u_B \land v_B \leq_t w_B\},\]
   for knowledge
   \[v_B \rightarrow w_B = \max_k\{u_B | u_B \land v_B \leq_k w_B\},\]
   that is,
   \[v_B \rightarrow w_B(A) = v_B(A) \rightarrow w_B(A),\]
   and
   \[(v_B \rightarrow w_B(A)) = v_B(A) \rightarrow w_B(A),\]
   for all \(A \in H_P\).
2. Duality operation: \(\partial w_B\) is pointwise defined by \(\partial w_B(A) = \partial(B(v_B(A)))\) for all \(A \in H_P\).
3. Knowledge negation: \(\neg v_B\) is pointwise defined by \((\neg v_B(A)) = (\neg v_B(A))\) for all \(A \in H_P\).

Proof: Let us prove that duality homomorphism holds for implications, \(v_B \rightarrow w_B = \partial(\partial v_B \rightarrow \partial w_B);\) for all other it is easy to verify in a pointwise fashion. So, we have that for all \(A \in H_P\) (consider that \(\partial\) is operator which preserves orderings):
\[\partial(v_B \rightarrow w_B(A)) = \partial(\max_k\{u_B | u_B \land v_B \leq_k w_B\})(A) = \partial(\max_k\{u_B(A) | (u_B \land v_B(A))(A) \leq_k w_B(A)\}) = \max_k\{\partial u_B(A) | (\partial u_B \land \partial v_B)(A) \leq_k \partial w_B(A)\} = \max_k\{\partial u_B(A) | (\partial u_B \land \partial v_B)(A) \leq_k \partial w_B(A)\} = (\partial u_B(A)) \rightarrow (\partial w_B(A)),\]
that is, \(\partial(v_B \rightarrow w_B) = \partial v_B \rightarrow \partial w_B\), thus, \(v_B \rightarrow w_B = \partial(\partial v_B \rightarrow \partial w_B)\).

Proposition 6: Let \(\Phi_q : B^{H_P} \rightarrow B^{H_P}\) be an “immediate consequence operator” for a logic program \(P\), monotonic w.r.t. the ordering \(\leq_q\), \(q \in \{t, k\}\), with a least fixpoint \(u = \Phi_q(u)\). Then there exists the “immediate consequence operator” \(\Phi_\tau = \Phi_\partial \Phi_q : B^{H_P} \rightarrow B^{H_P}\), of the dual program \(\partial P = \{\partial \varphi | \varphi \in P\}\), monotonic w.r.t. the dual ordering \(\leq_\partial\), with a least fixpoint \(u' = \partial u\).

Proof: By monotonicity of \(\Phi\) we have that \(v \leq_q w\) implies \(\Phi_q(v) \leq_q \Phi_q(w)\). So, by duality \(v' = \partial v \leq_\partial dw = w'\) and \(\partial \Phi_q(w) \leq_\partial \Phi_q(\partial w) = \Phi_\partial(w')\).
(1) \( v_{i+1}(A) = \tau_i((B_{11} \land .. \land B_{1k_1}) \lor .. \lor (B_{nk_1} \land .. \land B_{nk_n})). \)

We have that the dual grounded program is \( \partial \mathcal{P}_* = \{ \partial(A \leftarrow B_{j1} \land .. \land B_{jk_1}) \mid A \leftarrow B_{j1},..,B_{j1,k_1} \in \mathcal{P}_* \} = \{ \partial A \leftarrow \partial B_{j1} \land .. \land \partial B_{jk_1} \mid A \leftarrow \partial B_{j1},..,\partial B_{jk_1} \in \mathcal{P}_* \}. \) So, the dual immediate consequence operator \( \Phi_i = \partial \Phi_i \partial \) is monotonic with respect to the truth ordering, such that \( v'_{i+1} = \Phi_i(v'_i), \) \( i = 0,1,2,.. \) with \( v'_i \) the bottom truth member in \( B^H \) which assigns to all atoms in \( H \) the false value \( f \) of Belnap’s 4-valued bilattice.

Let us show that the next interpretation \( v'_{i+1} \) is determined as follows: for a ground atom \( A \), let \( S = \{ \partial A \leftarrow \partial B_{j1} \land .. \land \partial B_{jk_1} \} \) set be all classes in a grounded program \( \mathcal{P}_* \), then

\[
(2) \quad \tau'_i(\partial A) = \tau'_i((\partial B_{11} \land .. \land \partial B_{1k_1}) \lor .. \lor (\partial B_{nk_1} \land .. \land \partial B_{nk_n})).
\]

We can apply the operator \( \partial \) to both sides of the equation (1). For the left side we obtain

\[
\partial v_{i+1}(A) = \partial \Phi_i v_i(A) = \partial \Phi_i(\partial v'_i(\partial A)) = \Phi_i(\tau'_i(\partial A)) = \tau'_{i+1}(\partial A).
\]

While for the right side we obtain

\[
\partial \tau'_i((B_{11} \land .. \land B_{1k_1}) \lor .. \lor (B_{nk_1} \land .. \land B_{nk_n})) =
\]

\[
= \partial((v_i(B_{11}) \land .. \land v_i(B_{1k_1}) \lor .. \lor (v_i(B_{nk_1}) \land .. \land v_i(B_{nk_n})))) =
\]

\[
= (\partial v_i(B_{11}) \land .. \land \partial v_i(B_{1k_1}) \lor .. \lor (\partial v_i(B_{nk_1}) \land .. \land \partial v_i(B_{nk_n})) =
\]

\[
= (\partial v_i(B_{11}) \land .. \land \partial v_i(B_{1k_1}) \lor .. \lor (\partial v_i(B_{nk_1}) \land .. \land \partial v_i(B_{nk_n}))
\]

That is, we obtain (2).

IV. LOGIC PROGRAMMING AND PARAMETERIZED D-BILATTICES

By a logic program we mean a finite set of universally quantified clauses of the form \( \forall(A \leftarrow L_1 \land .. \land L_m), \) and a set of constraints \( \forall( \leftarrow L_1 \land .. \land L_m), \) where \( m \geq 0, A \) is an atom and \( L_i \) are positive or negative literals (see [20]). Following a standard convention, such clauses will be simply written as clauses of the form \( \forall(A \leftarrow L_1 \land .. \land L_m). \)

The alphabet of a program \( P \) consists of all constants, predicate and functional symbols that appear in \( P, \) a countably infinite set of variable symbols, connectives (\( \land, \lor, \neg, \leftarrow \) i.e., and, or, not and logic implication, respectively), and the usual punctuation symbols. We assume that if \( P \) does not contain any constant, then one is added to the alphabet. The language \( \mathcal{L} \) of \( P \) consists of all the well-formed formulae of the so obtained first order theory.

The intuitionistic logic is based on a Heyting algebra, where the conjunction/disjunction is equal to meet/join operations of a lattice, and implication is a relative pseudo-complement. The intuitionistic negation (i.e., pseudo-complement) is not good candidate for a logic programming based on bilattices because it is non monotonic w.r.t. both orderings, and, consequently, can not be used for computation of a fixpoint semantics for logic programs. In fact in logic programming we use default negation, which corresponds to the truth-negation in a bilattice: it is non monotonic w.r.t. the truth ordering but is monotonic w.r.t. the knowledge ordering, so that we can compute the least fixpoint semantics for many-valued logic programs w.r.t. the knowledge ordering.

In what follows we will extend such intuitionistic bilattice with the set of other conjunctive/disjunctive operators used for probabilistic strategies: the obtained extended D-bilattice algebra will be called 'parameterized D-bilattice’. As first step we introduce the basic parameterized Heyting algebra: based on it we will define different families of parameterized Dbilattices, that is, for belief and confidence-level based logics.

Definition 5: We define the parameterized Heyting algebra by the following extension of the Heyting algebra [21], \( \{L, 0, 1, \neg_w, \{\land_p \mid p \in Par\}, \{\lor_p \mid p \in Par\}\}, \) where \( L = \{0,1\} \) is interval of reals from 0 to 1. \( \neg_w \) is a unique negation such that \( \neg_w x = 1 - x, \) and \( Par = \{w, s, m, e\} \) are parameters, such that for any pair \( x, y \in L \)

1. weak conjunction, \( x \land_w y = \min\{x, y\} \) (meet operation in a lattice \( L )
2. strong conjunction, \( x \land_s y = \max\{0, x + y - 1\} \)
3. multiplicative conjunction, \( x \land_m y = x \cdot y \)
4. mutually exclusive conjunction, \( x \land_e y = 0. \)

The parametric disjunctions are defined by de Morgan laws \( x \lor_p y = \neg_{w}(\neg_{w} x \land p \neg_{w} y), p \in Par, \) and the parametric implications by relative pseudo complements \( y \Rightarrow_p x = \max\{z \in L \mid x \land_p z \leq y\}, \) for any \( p \in Par. \)

A. Belief based logic

Suppose we are attempting to assign, not classical truth values, but probability estimates to formulae, that is, the closed intervals \( [a, b] \subseteq [0,1] \). Such truth values can be ordered by set inclusion (inverse to knowledge ordering), thus we set \( [a, b] \leq_k [c, d] \) if \( a \leq c \) and \( d \leq b, \) that is, \( [c, d] \subseteq [a, b] \) if we consider that \( [c, d] \) is empty set if \( c > d \), with the bottom element \( 0_K = [0,1] \) and the top element \( 1_K = [1,0] \). Such generalized truth values should be partially ordered by degree of truth, that is \( [a, b] \leq_k [c, d] \) if \( a \leq c \) and \( b \leq d, \) with the bottom element \( 0_T = [0,0] \) and the top element \( 1_T = [1,1] \). Thus we obtain the interval-based bilattice with set of truth values \( L_B = \{[a, b] \mid 0 \leq a, b \leq 1\} \). Differently from Fitting, in [22], and Ginsberg, in [6], we will consider all values in \( L_B \) (also ‘inconsistent’ values \( [a, b] \) with \( a > b \). It corresponds to a simple way of constructing bilattices, due to Ginsberg, as the structure \( < L_1 \times L_2, \leq_{1,2}, k >, \) where, in this case \( L_1 = L_2 = \) a complete lattice of reals from 0 to 1, with classic orderings \( \leq \) of real numbers. Think of \( L_1 \) as the lattice of values used to measure the lower boundary degree of belief we have in a sentence; probabilities, weighted opinions of experts, etc., and of \( L_2 \) as the lattice of values used to measure the upper boundary degree of belief. Then members of \( L_1 \times L_2 \) embodies an assessment of belief degree. The ordering \( \leq_k \) intuitively says that degree of truth increases if both belief boundaries increase (in [9] for example). Likewise \( \leq_k \) intuitively says degree of knowledge increases if the precision goes up, in the case of ‘consistent’ intervals (used in [23]).

Proposition 7: Let \( (L, 0, 1, \land_w, \Rightarrow_w, \neg_w) \) be the interval-based extended Heyting algebra with \( L = [0,1]. \) The interval-
based bilattice is a D-bilattice composed by two lattices, a t-lattice \((L_B, \alpha_t)\), with \(\alpha_t = \{\land, \lor, \neg, \partial\}\), and a k-lattice \((L_B, \alpha_k)\), with \(\alpha_k = \{\otimes, \land, \neg, \partial\}\), such that:

1. Joint operation for \(\leq i\) ordering is the positive correlation probability strategy conjunction \([x, y] \wedge [x_1, y_1] = [x \wedge_u x_1, y \wedge_w y_1]\).

2. Intuitionistic implication:
   \[\hat{x, y} \rightarrow \hat{x_1, y_1} = \hat{x \Rightarrow u x_1, y \Rightarrow w y_1}\].

3. Truth negation:
   \[\neg [x, y] = [\neg_w y, \neg_u x] = [1 - y, 1 - x]\].

4. Knowledge negation:
   \[\partial [x, y] = [x, \neg y] = [x, 1 - y]\].

**Proof:** Easy to verify. \(\square\)

The join and meet operation in this D-bilattice correspond to the positive conjunction/disjunction probabilistic strategies. We can extend duality principle also for other conjunction/disjunction strategies, as ignorance, negative correlation, independence and mutually-exclusive probabilistic strategies.

For a given parameterized Heyting algebra over single probability values in \(L = [0, 1]\), we are able to define the belief-based logic, parameterized by a number of possible probabilistic strategies, over probability-interval based bilattice \(L_B\) (obviously, holds that \(L_B = L \times L\)).

**Proposition 8:** The interval-based parameterized bilattice is a D-bilattice composed by two lattices, a t-lattice \((L_B, \alpha_t)\), with \(\alpha_t = \{\land, \lor, \neg, \partial\}\), and a k-lattice \((L_B, \alpha_k)\), with \(\alpha_k = \{\otimes, \land, \neg, \partial\}\), and the parametric implications by relative pseudo complements \([x, y] \rightarrows_p [x_1, y_1] = \max\{\{\land, \lor, \neg, \partial\}[x, y] \wedge \land_p \neg [x_1, y_1]\}\), for any \(p \in \text{Par}_B\), and the parametric implications by relative pseudo complements \([x, y] \leftarrows_p [x_1, y_1] = \max\{[x \wedge \land_p \neg [x_1, y_1], y \wedge \lor_p \neg [x_1, y_1]\}\}.

5. Mutually-exclusive implication, \([x, y] \rightarrow_{\text{ce}} [x_1, y_1] = [x \Rightarrow_e x_1, y \Rightarrow_e y_1] = 1_T\). It is easy verify that knowledge conjunctions/disjunctions satisfy deduction laws w.r.t. the knowledge negation, and that holds \(\neg = \partial = \partial\). \(\square\)

For the interval-probability logic programming, the parameterized implications can be used for definitions of satisfaction relation for rules of such probabilistic logic programs: each of them preserves the principle of truth conservation from body into the head of rules, so that can be used in each fixed point algorithm for computation of models for these logic programs. It is interesting that the strong intuitionistic negation \(\neg_{nc}[x, y] = [x, y] \Rightarrow_{nc} [0, 0] = [1 - x, 1 - y]\), such that \(\neg_{nc} = \neg\), plays a central role as default negation operator in Paraconsistent logic programming [24] able to deal with both uncertain and contradiction information.

**B. Confidence-level based logic**

Let us consider now the case when we are dealing with incomplete knowledge, where different evidence may contradict one other, so that is convenient to explicitly representing both belief and doubt (differently from interval based valuations, where the doubt is just equal to the epistemic negation (complement) of the belief), as it was argued in [10]. In this case our lattice is the set of pairs of intervals, that is, a cartesian product type \(L_C = L_B \times L_B\), so that each of its members \([\{x, y\}, [z, v]\] \in L_C\) is a confidence level, that is a couple of a belief \([x, y]\) and a doubt \([z, v]\), which we associate to the facts of our uncertain and incomplete knowledge base. Such confidence level is consistent [9] if \(x + z \leq 1\).

In this lattice the truth ordering \(\leq_c\) which increases the belief and decreases the doubt of facts is used to define the monotonic fixpoint operator for model theoretic characterization of confidence level logic programs (see [6], [9] for more details), that is \([\{x, y\}, [z, v]\] \leq_c ([x_1, y_1], [z_1, v_1])\).

The join (conjunction) and the meet operation (disjunction) for this truth ordering \(\leq_c\) and the epistemic negation \(\neg_c\), which reverses this truth ordering, of this lattice \(L_C\) are defined by Ginsberg [6], with \(\neg_c([x, y], [z, v]) = ([z, v], [x, y])\).

The bottom and the top truth members of this lattice \(L_C\) are \(0_C = ([0, 0], [1, 1])\), and \(1_C = ([1, 1], [0, 0])\) if \(x + z \leq 1\).

The knowledge conjunction/disjunction strategies and implications are defined for each \(p \in \text{Par}_B\), by \([x, y] \otimes_p [x_1, y_1] = [\neg [x, y] \land \neg [x_1, y_1]]\), and \([x, y] \otimes_p [x_1, y_1] = [\neg [x, y] \lor \neg [x_1, y_1]]\).

The join (conjunction) and the meet operation (disjunction) for this knowledge ordering \(\leq_k\) and the knowledge negation \(\neg_k\), which reverses this knowledge ordering, are defined by \(\neg_k([x, y], [z, v]) = ([z, v], [x, y]) = ([1 - v, 1 - z], [1 - y, 1 - x])\) if \([x, y] \leq_k [x_1, y_1] and [z, v] \leq_k [z_1, v_1]\).

The join operation (conjunction) and the meet operation (disjunction) for this knowledge ordering \(\leq_k\) and the knowledge negation \(\neg_k\), which reverses this knowledge ordering, are defined by \(\neg_k([x, y], [z, v]) = ([z, v], [x, y]) = ([1 - v, 1 - z], [1 - y, 1 - x])\). The bottom and the top knowledge members of this lattice \(L_C\) are \(0_K = ([0, 0], [0, 0])\), and \(1_K = ([1, 1], [1, 1])\).
that for any pair \( (x, y), [z, v], ([x_1, y_1], [z_1, v_1]) \in L_C \), \( p \in Par_B \):

1. The parametric conjunctions are defined by
   \[(x, y), [z, v] \land_{C_p} ([x_1, y_1], [z_1, v_1]) =
   (x, y) \land_p [x_1, y_1], [z, v] \lor_p [z_1, v_1]).\]

2. The parametric disjunctions are defined by de Morgan laws
   \[(x, y), [z, v] \lor_{C_p} ([x_1, y_1], [z_1, v_1]) =
   (x, y) \lor_p [x_1, y_1], [z, v] \land_p [z_1, v_1]).\]

3. The parametric implications by relative pseudo completions
   \[(x, y), [z, v] \Rightarrow_{C_p} ([x_1, y_1], [z_1, v_1]) =
   \max\{[a, b], [c, d] \in L_C | ([x, y], [z, v] \land_{C_p} ([a, b], [c, d]) \leq C_p([x_1, y_1], [z_1, v_1])\} =
   ([x, y] \rightarrow_p [x_1, y_1]); \gamma = (\neg[z, v] \rightarrow_p \neg[z_1, v_1])\] for every \( p \in Par_B \).

4. Duality operation: \( \partial_{C_p} ([x, y], [z, v]) = ([x, y], [\neg z, v]) \).

5. The knowledge conjunction/disjunction strategies and
   implications are defined, for each \( p \in Par_B \), by
   \[(x, y), [z, v] \otimes_{C_p} ([x_1, y_1], [z_1, v_1]) =
   \partial_{C_p}([x, y], [z, v]) \land_{C_p} \partial_{C_p}([x_1, y_1], [z_1, v_1]),\]
   \[(x, y), [z, v] \oplus_{C_p} ([x_1, y_1], [z_1, v_1]) =
   \partial_{C_p}([x, y], [z, v]) \lor_{C_p} \partial_{C_p}([x_1, y_1], [z_1, v_1]),\]
   \[(x, y), [z, v] \Rightarrow_{C_p} ([x_1, y_1], [z_1, v_1]) =
  \partial_{C_p}([x, y], [z, v]) \Rightarrow_{C_p} \partial_{C_p}([x_1, y_1], [z_1, v_1]).\]

Proof: It is easy to verify. In fact, given \( \alpha = ([x, y], [z, v]), \beta = ([x_1, y_1], [z_1, v_1]) \), then, for conjunctions (and disjunctions) the proof is equal to the Th.4.1 in [9], that is \( \alpha \land_{C_p} \beta = 1 \).

1. Positive correlation (join operation),
   \([(\min \{x, x_1\}, \min \{y, y_1\}), [\max \{z, z_1\}, \max \{v, v_1\}]).\]

2. Ignorance,
   \([(\max \{0, x + x_1 - 1\}, \min \{y, y_1\}), [\max \{z, z_1\}, \min \{1, v + v_1\})].\]

3. Negative correlation strategy conjunction,
   \([(\min \{0, x + x_1 - 1\}, \max \{y, y_1 + 1\}), \min \{1, z + z_1\}, \min \{1, v + v_1\}).\]

4. Independence,
   \([(x, x_1, y, y_1), [z + z_1 - v + v_1], v + v_1 - v_1]).\]

5. \( 0_{C_p} \), for mutually-exclusive strategy conjunction. \( \square \)

V. Conclusion

In this paper we defined a family of intuitionistic
truth/knowledge symmetric bilattices composed as couples of
parameterized Heyting algebras \( L, L_B \) and \( L_C \), downward compatible, based on complete distributive lattices with truth ordering and the epistemic negation which reverse such ordering. Parameterizations of these symmetric bilattices is given for a number of possible conjunction probability strategies which can be used for the rules in probabilistic logic programming, and which are a kind of more sceptic algebraic approximations for a join operation of these lattices (the basic conjunctive strategy). By connection with a paraconsistent logic programming [24] we are able to extend the use of these parameterized intuitionistic logics to deal with both uncertain and inconsistent information. Moreover, we are able to extend such logic programming also with nested implications: for example by allowing the formulae with implication also in bodies of programming rules.

This coherent and incremental approach to different probabilistic approaches for modeling uncertainty gives a clear light on the nature of deep interdependency between different measure types for uncertainty, and different conjunctive/disjunctive strategies. Such bottom-up development strategy is well suited for the generalized fuzzy sets and fuzzy logic and extends their basic techniques for bilattice-based logic programs.

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