Autoreferential semantics for many-valued modal logics

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ABSTRACT. *In this paper we consider the class of truth-functional modal many-valued logics with the complete lattice of truth-values. The conjunction and disjunction logic operators correspond to the meet and join operators of the lattices, while the negation is independently introduced as a hierarchy of antitonic operators which invert bottom and top elements. The non-constructive logic implication will be defined for a subclass of modular lattices, while the constructive implication for distributive lattices (Heyting algebras) is based on relative pseudo-complements as in intuitionistic logic. We show that the complete lattices are intrinsically modal, with banal identity modal operator. We define the autoreferential set-based representation for the class of modal algebras, and show that the autoreferential Kripke-style semantics for this class of modal algebras is based on the set of possible worlds equal to the complete lattice of algebraic truth-values. The philosophical assumption is based on the consideration that each possible world represents a level of credibility, so that only propositions with the right logic value (i.e., level of credibility) can be accepted by this world, then we connect it with paraconsistent properties and LFI logics. The bottom truth value in this complete lattice corresponds to the trivial world in which each formula is satisfied, that is, to the world with explosive inconsistency. The top truth value corresponds to the world with classical logics, while all intermediate possible worlds represent the different levels of paraconsistent logics.*

KEYWORDS: *many-valued logics, modal logics, Kripke-style semantics, paraconsistency.*

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1. Introduction

A significant number of real-world applications in Artificial Intelligence have to deal with partial, imprecise and uncertain information, and that is the principal reason

for introducing non-classic truth-functional many-valued logics, as for example, fuzzy, bilattice-based and paraconsistent logics, etc.

All these logics use the conjunction and disjunction logic operators as meet and join lattice operators with truth partial ordering between the set of truth values. The class of complete lattices of truth values is the most significant for logics, just because each finite set of truth values, or totally ordering in infinite set of values (as in fuzzy logic) is a *complete* lattice.

Thus, the many-valued modal logics we consider have a natural algebraic semantics based on a complete lattice of truth values, extended by other algebraic operators: negation, implication for a subclass of modular lattices, relative pseudo-complement (for distributive lattices) (Jipsen *et al.*, 2002) for intuitionistic many-valued implication, and unary normal modal operators (the first formal semantics for modal logic was based on many-valuedness, proposed by Łukasiewicz in 1918–1920, and consolidated in 1953 in a 4-valued system of modal logic (Łukasiewicz, 1953)).

There are strong links between the structural rules the logics satisfy and the properties of their *algebraic* models (which are, in the case of predicate based logic, the many-valued Herbrand models). The distinctive feature of algebraic semantics is that it is truth-functional. Besides algebraic models also Kripke-style or *relational* models exist. The distinctive feature of Kripke-style semantics is that accessibility relations over possible worlds are used in the definition of satisfiability, which is not just a mechanical truth-functional translation of the logic formula into the model.

There is a lot of work dedicated to this duality for lattices (Urquhart, 1978) with operators. A good recent overview can be found in (Brink *et al.*, 1993; Clark *et al.*, 1998; Haim, 2000; Sofronie-Stokkermans, 2003).

Differently from their general approach, we introduce the *autoreferential duality* for complete lattices: we use (autoreferentially) as a set of possible worlds for a Kripke frame the underlying complete lattice of truth values. This is a particular case where the frame is a partial order, as for example in (Celani *et al.*, 1997). Consequently, the main contribution of this paper is a general way of establishing links between algebraic and Kripke-style semantics for the class of many-valued modal logics, based on this autoreferential duality, and the consequent paraconsistent properties.

The plan of this paper is the following: After a brief introduction into the modal predicate logics, and a short introduction into complete lattices and modal truth-functional algebraic logics based on Galois connections for modal operators, in Section 2 we present the *autoreferential semantics* for complete lattices and their modal extensions. The other original contributions to this modal algebraic theory are given in the following three Sections: In Section 3 we develop a general Representation Theorem for the class of logics based on complete lattices with a hierarchy of negation operators, by using the Dedekind-McNeile Galois connection for the partial order relation of the lattice and on the derived closure operator. The obtained set-based isomorphic algebra has the usual set intersection as meet operator, but the join operator is a generalization of the set union: we also introduce the subset of complete

lattices, where the join corresponds exactly to the set union. In Section 4 we enrich this compact lattice representation by introducing intuitionistic implication (when a complete lattice is distributive), and a set of modal operators and modal negations. In Section 5 we introduce also the autoreferential world-based Kripke style definition for a many-valued modal logics, with the Kripke frame based on a set of possible worlds equal to the complete lattice of truth values, and show that it is equivalent to standard algebraic semantics based on many-valued Herbrand models. Finally, in Section 6 we discuss the relationships between many-valuedness, autoreference and paraconsistency for this class of many-valued logics.

In what follows we will use the algebraic meet and join symbols of a complete lattice $(X, \leq)$, where X is a set of algebraic truth-values and $\leq$ a partial order in X , also as symbols for corresponding *logic* connectives, conjunction and disjunction, while for other connectives we will maintain the formal distinction.

1.1. Introduction to multi-modal predicate logic

Given two sets A and B , we denote by A^B , the set of all functions from B to A , and by $A^n, n \geq 1$ the n -th cartesian product $A \times \dots \times A$.

We denote by $\mathcal{M} = (\mathcal{W}, \{\mathcal{R}_i \mid i \geq 1\}, S, V)$ the multi-modal Kripke model with: the set of possible worlds $\mathcal{W}$, accessibility relations $\mathcal{R}_i \subseteq \mathcal{W} \times \mathcal{W}$, non empty set of individuals S , and V is a function defined in the following two cases:

- 1) $V : \mathcal{W} \times F \rightarrow \bigcup_{n < \omega} S^{S^n}$, with F a set of functional symbols of the language, such that for any world $w \in \mathcal{W}$, $f \in F$, $V(w, f) : S^n \rightarrow S$ is a function (interpretation of f in w).
- 2) $V : \mathcal{W} \times P \rightarrow \bigcup_{n < \omega} \mathbf{2}^{S^n}$, with P a set of predicate symbols of the language, $\mathbf{2} = \{0, 1\}$ set of logic values (0:false, 1:true), such that for any world $w \in \mathcal{W}$, $r \in P$, the function $V(w, r) : S^{arity(r)} \rightarrow \mathbf{2}$ defines the extension of r in a world w , $\{\mathbf{d} \mid \mathbf{d} \in S^{arity(r)}, V(w, r)(\mathbf{d}) = 1\}$, where $\mathbf{d}$ denotes a tuple of constants in S .

We define the set of terms of this modal logic as follows: all variables $x \in Var$, and constants $d \in S$ are terms; if $f \in F$ is a functional symbol of arity n and $t_1, \dots, t_n$ are terms, then $f(t_1, \dots, t_n)$ is a term.

An atom is the formula $r(t_1, \dots, t_n)$ where $r \in P$ is a predicate symbol with arity n , and $t_i, 1 \leq i \leq n$ are the terms. Any atom is a logic formulae. The combination of logic formulae by logic connectives (conjunction $\wedge$, disjunction $\vee$, negation $\neg$, implication $\Rightarrow$, existential modal operators $\Diamond_i$, etc..) is another logic formula.

The extension of a logic formula φ , w.r.t a model $\mathcal{M}$, a world $w \in \mathcal{W}$, and assignment $g : Var \rightarrow \bigcup_{n < \omega} S^n$ is denoted by $\|\varphi\|^{\mathcal{M}, w, g}$. Thus, if $r \in F \cup P$ then $\|r\|^{\mathcal{M}, w, g} = V(w, r)$, that is, for a set of terms $t_1, \dots, t_n$ where n is arity of r , $\|r(t_1, \dots, t_n)\|^{\mathcal{M}, w, g} = V(w, r)(\|t_1\|^{\mathcal{M}, w, g}, \dots, \|t_n\|^{\mathcal{M}, w, g})$, and, if t is a variable then $\|t\|^{\mathcal{M}, w, g} = g(t)$.

For any formula φ we define $\mathcal{M} \models_{w, g} \varphi$ iff $\|\varphi\|^{\mathcal{M}, w, g} = 1$.

The semantics for the universal modal operator $\Diamond_i$, applied to a formula ϕ , is defined by: $\mathcal{M} \models_{w,g} \Diamond_i \phi$ iff $\exists w' ((w, w') \in \mathcal{R}_i \text{ and } \mathcal{M} \models_{w',g} \phi)$. The universal modal operator $\Box_i$ is equal to $\neg \Diamond_i \neg$.

A formula ϕ is said to be *true in a model* $\mathcal{M}$ if $\mathcal{M} \models_{w,g} \phi$ for each assignment function g and possible world w . A formula is said to be *valid* if it is true in each model. With ϕ/g we denote a formula (with variables) ϕ for a given assignment g , while $\|\phi/g\| = \{w \mid \mathcal{M} \models_{w,g} \phi\}$ is the set of worlds where the ground formula ϕ/g is satisfied.

Generally normal 2-valued modal logics are not truth-valued (there is no homomorphism between the syntactical structure of the logic language and the Boolean algebra (*i.e.*, distributive lattices with complements) of truth functions $(\mathbf{2}, \leq)$, because we have only four unary functions in $\mathbf{2}^2$: identity, negation, tautology and contradiction. Thus the existential unary modal operator $\Diamond_i$, and its dual universal modal operator $\Box_i$, can not be represented as the function from $\mathbf{2}$ into $\mathbf{2}$, and it was the reason why Jan Łukasiewicz introduced around 1918 more truth-values than the ordinary two.

In fact, there is always the equivalent *truth-functional* many-valued logic, with the set of higher-order truth-values (Majkić, 2006a) in the Boolean-lattice functional space $(\mathbf{2}^{\mathcal{W}}, \leq, \wedge, \vee, \neg)$, isomorphic to the powerset Boolean algebra $(\mathcal{P}(\mathcal{W}), \subseteq, \cap, \cup, -)$, *i.e.*, there is the isomorphism of Boolean algebras $im : (\mathbf{2}^{\mathcal{W}}, \leq, \wedge, \vee, \neg) \simeq (\mathcal{P}(\mathcal{W}), \subseteq, \cap, \cup, -)$, such that for any two functional algebraic values $f, g \in \mathbf{2}^{\mathcal{W}}$, $f \leq g$ iff $im(f) \subseteq im(g)$, where $im(f) = \{w \in \mathcal{W} \mid f(w) = 1\}$ is the image of f .

The meet, join and negation operations in the Boolean algebra $(\mathbf{2}^{\mathcal{W}}, \leq, \wedge, \vee, \neg)$ are defined by this isomorphism as :

$$\begin{aligned} h &= f \wedge g : \mathcal{W} \rightarrow \mathbf{2}, \text{ such that } im(h) = im(f) \cap im(g), \\ h &= f \vee g : \mathcal{W} \rightarrow \mathbf{2}, \text{ such that } im(h) = im(f) \cup im(g), \\ h &= \neg f : \mathcal{W} \rightarrow \mathbf{2}, \text{ such that } im(h) = -im(f), \text{ where } - \text{ is the set complement in } \mathcal{W}. \end{aligned}$$

The logic implication in this Boolean algebra is defined classically by $f \Rightarrow g =_{def} \neg f \vee g$. The truth-functional universal (multiplicative) modal operator $\Box_i : \mathbf{2}^{\mathcal{W}} \rightarrow \mathbf{2}^{\mathcal{W}}$ for normal Kripke logic with the binary accessibility relation $\mathcal{R}_i$ is defined by: $h = \Box_i f : \mathcal{W} \rightarrow \mathbf{2}$, such that $im(h) = \{w \mid \forall w' ((w, w') \in \mathcal{R}_i \text{ implies } w' \in im(f))\}$.

So that, a *non* truth-functional 2-valued modal logic is equivalently represented by a *many-valued truth-functional* algebraic logic based on Boolean algebra extended by normal modal algebraic operators $(X, 0, 1, \leq, \wedge, \vee, \neg, \Box_i)$, where $X = \mathbf{2}^{\mathcal{W}}$ and $0, 1 \in X$ are the bottom and top value in X , with also an infinite number of algebraic truth-values (Dugundji, 1940). The same approach can be applied to any many-valued *non* truth-functional modal logic, thus it is enough to analyze only the *algebraic truth-functional* many-valued modal logics, as is done in the rest of this paper. Vice versa,

any algebraic many-valued logic program, based on the complete lattice of truth-values, can be equivalently represented by a non truth-functional 2-valued multi-modal logic (Majkić, 2006e) with ternary accessibility relations.

Given a modal algebra $(X, 0, 1, \leq, \wedge, \vee, \neg, \Box_i)$ the standard relational semantics based on Stone's representation theorem, is represented by a descriptive general frame introduced by Goldblatt (Goldblatt, 1976a; Goldblatt, 1976b; Bull *et al.*, 1984), where a possible world is an ultrafilter F defined as a subset of X which satisfies the following conditions:

- $1 \in F$ and not $0 \in F$
- if $x, y \in F$ then $x \wedge y \in F$
- if $x \in F$ and $x \leq y$ then $y \in F$
- for each $x \in X$, either $x \in F$ or $\neg x \in F$.

Instead of this standard relational semantics for modal logic, where a possible world is a *subset* of elements in X (an ultrafilter), in this paper we will use an more simple *autoreferential* semantics where a possible world is an *element* in X .

1.2. Introduction into lattice algebras and their extensions

Posets and lattices (posets such that for all its elements x and y , the set $\{x, y\}$ has both a join (lub—least upper bound) and a meet (glb—greatest lower bound)) with a partial order $\leq$ play an important role in what follows. A *bounded* lattice has a greatest (top) and least (bottom) element, denoted by convention by 1 and 0 .

Finite meets in a poset will be written as $\bigwedge$, and finite joins as $\bigvee$. A lattice (poset) X is *complete* if each (also infinite) subset $S \subseteq X$ (or, $S \in \mathcal{P}(X)$ where $\mathcal{P}$ is the symbol for powerset, and $\emptyset \in \mathcal{P}(X)$ denotes the empty set) has the least upper bound (supremum) denoted by $\bigvee S \in X$ (when S has only two elements the supremum corresponds to the join operator $\vee$). Each finite bounded lattice is a complete lattice. Each subset S has the greatest lower bound (infimum) denoted by $\bigwedge S \in X$, given as $\bigvee \{x \in X \mid \forall y \in S. x \leq y\}$. The complete lattice is bounded and has the bottom element, $0 = \bigwedge X \in X$, and the top element $1 = \bigvee X \in X$.

A function $l : X \rightarrow Y$ between posets X, Y is *monotone* if $x \leq x'$ implies $l(x) \leq l(x')$ for all $x, x' \in X$. Such a function $l : X \rightarrow Y$ is said to have a right (or upper) adjoint if there is a function $r : Y \rightarrow X$ in the reverse direction such that $l(x) \leq y$ iff $x \leq r(y)$ for all $x \in X, y \in Y$. Such a situation forms a Galois connection and will often be denoted by $l \dashv r$. Then l is called a left (or lower) adjoint of r . If X, Y are complete lattices (posets) then $l : X \rightarrow Y$ has a right adjoint iff l preserves all joins (it is *additive*, i.e., $l(x \vee y) = l(x) \vee l(y)$ and $l(0_X) = 0_Y$ where $0_X, 0_Y$ are bottom elements in complete lattices X and Y respectively). The right adjoint is then $r(y) = \bigvee \{z \in X \mid l(z) \leq y\}$. Similarly, a monotone function $r : Y \rightarrow X$ is a right adjoint (it is *multiplicative*, i.e., has a left adjoint) iff r preserves all meets; the left adjoint is then $l(x) = \bigwedge \{z \in Y \mid x \leq r(z)\}$.

Each monotone function $l : X \rightarrow Y$ on a complete lattice (poset) X has both a *least* fixed point $\mu l \in X$ and *greatest* fixed point $\nu l \in X$. these can be described explicitly as: $\mu l = \bigwedge \{x \in X \mid l(x) \leq x\}$ and $\nu l = \bigvee \{x \in X \mid x \leq l(x)\}$.

In what follows we denote by $y < x$ iff ($y \leq x$ and not $x \leq y$), and we denote by $x \bowtie y$ two unrelated elements in X (so that not ($x \leq y$ or $y \leq x$)). An element in a lattice $x \neq 0$ is a *join-irreducible* element iff $x = a \vee b$ implies $x = a$ or $x = b$ for any $a, b \in X$. An element in a lattice $x \in X$ is an *atom* iff $x > 0$ and $\nexists y. (x > y > 0)$.

A lower set (down closed) is any subset Y of a given poset $(X, \leq)$ such that, for all elements x and y , if $x \leq y$ and $y \in Y$ then $x \in Y$.

A *Heyting algebra* is a bounded lattice X with finite meets and joins such that for each element $x \in X$, the function $(_) \wedge x : X \rightarrow X$ has a right adjoint $x \multimap (_)$, also called an algebraic implication. An equivalent definition can be given by considering a bounded lattice such that for all x and y in X there is a greatest element z in X , denoted by $x \multimap y$, such that $z \wedge x \leq y$, i.e., $x \multimap y = \bigvee \{z \in X \mid z \wedge x \leq y\}$ (relative pseudo-complement). In Heyting algebra we can define negation $\neg x$ as a pseudo-complement $x \multimap 0$. Then $x \leq \neg \neg x$. A complete Heyting algebra is a Heyting algebra which is complete as a poset. A complete lattice is thus a complete Heyting algebra iff the following *distributivity* $x \wedge (\bigvee S) = \bigvee_{y \in S} (x \wedge y)$ holds.

The negation and implication operators can be represented as the following monotone functions: $\neg : X \rightarrow X^{OP}$ and $\Rightarrow : X \times X^{OP} \rightarrow X^{OP}$, where X^{OP} is the lattice with inverse partial ordering and $\wedge^{OP} = \vee, \vee^{OP} = \wedge$.

The smallest complete distributive lattice is denoted by $\mathbf{2} = \{0, 1\}$ with classic two values, false and true respectively. It is also complemented Heyting algebra, consequently it is Boolean. A *Galois algebra* is a complete Heyting algebra both with a "nexttime" monotone function $\text{From} : X \rightarrow X$ that preserves all meets (i.e., right adjoint). Such Galois algebras are often called Heyting algebras with (unary) modal operators.

1.3. Introduction to sequents and bivaluations

Sequent calculus was developed by Gentzen (Gentzen, 1932), inspired by some ideas of Paul Herz (Hertz, 1929). Given a propositional logic language $\mathcal{L}$ (set of logic formulae) a sequent is a consequence pair of formulae $s = (\phi; \psi) \in \mathcal{L} \times \mathcal{L}$, denoted also by $\phi \vdash \psi$.

A Gentzen system, denoted by a pair $\mathcal{G} = \langle \mathbb{L}, \Vdash \rangle$ where $\Vdash$ is a finitary consequence relation on a set of sequents in $\mathbb{L} \subseteq \mathcal{L} \times \mathcal{L}$, is said to be *normal* if it satisfies the following conditions: for any sequent $s = \phi \vdash \psi \in \mathbb{L}$ and the set of sequents $\Gamma = \{s_i = \phi_i \vdash \psi_i \in \mathbb{L} \mid i \in I\}$,

- 1) (reflexivity) if $s \in \Gamma$ then $\Gamma \Vdash s$
- 2) (transitivity) if $\Gamma \Vdash s$ and for every $s' \in \Gamma$, $\Theta \Vdash s'$, then $\Theta \Vdash s$

- 3) (finiteness) if $\Gamma \Vdash s$ then there is finite $\Theta \subseteq \Gamma$ such that $\Theta \Vdash s$.
 4) for any homomorphism σ from $\mathbb{L}$ into itself (*i.e.*, substitution), if $\Gamma \Vdash s$ then $\sigma[\Gamma] \Vdash \sigma(s)$, *i.e.*, $\{\sigma(\phi_i) \vdash \sigma(\psi_i) \mid i \in I\} \Vdash (\sigma(\phi) \vdash \sigma(\psi))$.

Notice that from (1) and (2) we obtain this monotonic property:

- 5) if $\Gamma \Vdash s$ and $\Gamma \subseteq \Theta$, then $\Theta \Vdash s$.

We denote by $C : \mathcal{P}(\mathbb{L}) \rightarrow \mathcal{P}(\mathbb{L})$ the closure operator such that $C(\Gamma) =_{\text{def}} \{s \in \mathbb{L} \mid \Gamma \Vdash s\}$, with the properties: $\Gamma \subseteq C(\Gamma)$ (from reflexivity (1)); it is monotonic $\Gamma \subseteq \Gamma_1$ implies $C(\Gamma) \subseteq C(\Gamma_1)$ (from (5)), and an involution, $C(C(\Gamma)) = C(\Gamma)$ also. Thus we obtain that

- 6) $\Gamma \Vdash s$ iff $s \in C(\Gamma)$.

Any sequent theory $\Gamma \subseteq \mathbb{L}$ is said to be a *closed* theory iff $\Gamma = C(\Gamma)$. This closure property corresponds to the fact that $\Gamma \Vdash s$ iff $s \in \Gamma$.

Each sequent theory Γ can be considered as a bivaluation (characteristic function) $\beta : \mathbb{L} \rightarrow \mathbf{2}$ such that for any sequent $s \in \mathbb{L}$, $\beta(s) = 1$ iff $s \in \Gamma$.

2. Many-valued model-theoretic autoreferential semantics

Let $\mathcal{L}$ be a propositional logic language obtained as free algebra, from the connectives in Σ of an algebra based on a complete lattice $(X, \leq)$ of algebraic truth-values (for example meet and join $\{\wedge, \vee\} \subseteq \Sigma$ are binary operators, negation $\neg \in \Sigma$ and other modal operators are unary operators, while each $x \in X \subseteq \Sigma$ is a constant (nullary operator)) and a set Var of propositional variables (letters) denoted by $p, r, q, \dots$. We will use letters $\phi, \psi, \dots$ for the formulae of $\mathcal{L}$. We define a (many-valued) *valuation* v as a mapping $v : \mathcal{L} \rightarrow X$ (notice that $X \subseteq \mathcal{L}$ are the constants of this language and we will use the same symbols as for elements of the lattice X), which is an homomorphism (for example, for any $p, q \in Var$, $v(p \odot q) = v(p) \odot v(q)$, $\odot \in \{\wedge, \vee, \Rightarrow\}$ and $v(\neg p) = \neg v(p)$, where $\wedge, \vee, \Rightarrow, \neg$ are conjunction, disjunction, implication and negation respectively) and is an identity for elements in X , that is, for any $x \in X$, $v(x) = x$. We denote by $\mathbb{V}_m$ the strict subset in $X^{\mathcal{L}}$ of all homomorphic *many-valued valuations*.

Then we define the following lattice-based consequence binary relation $\vdash \subseteq \mathcal{L} \times \mathcal{L}$ between the formulae (it is analog to the binary consequence system from (Dunn, 1995) for the distributive lattice logic *DLL*), where each consequence pair $\phi \vdash \psi$ is a *sequent* also.

EXAMPLE 1. — Let us consider the Distributive modal logic (Dunn, 1995) (with $\Box$ universal modal operator, and its left adjoint existential modal operator $\Diamond$, with $\Diamond \dashv \Box$) and with a negative modal operator $\neg$. The Gentzen-like system $\mathcal{G}$ of this logic language $\mathcal{L}$ contains the following axioms (sequents) and rules:

(AXIOMS) $\mathcal{G}$ contains the following sequents:

- 1) $\phi \vdash \phi$ (reflexive)
- 2) $\phi \vdash 1, 0 \vdash \phi$ (top/bottom axioms)
- 3) $\phi \wedge \psi \vdash \phi, \phi \wedge \psi \vdash \psi$ (product projections: axioms for meet)
- 4) $\phi \vdash \phi \vee \psi, \psi \vdash \phi \vee \psi$ (co-product injections: axioms for join)
- 5) $\phi \wedge (\psi \vee \varphi) \vdash (\phi \wedge \psi) \vee (\phi \wedge \varphi)$ (distributivity axiom)
- 6) $\Box(\phi \wedge \psi) \vdash \Box\phi \wedge \Box\psi, 1 \vdash \Box 1$ (multiplicative modal property axioms)
- 7) $\Diamond(\phi \vee \psi) \vdash \Diamond\phi \vee \Diamond\psi, \Diamond 0 \vdash 0$ (additive modal property axioms)
- 8) $\neg\phi \wedge \neg\psi \vdash \neg(\phi \vee \psi), 1 \vdash \neg 0$ (additive modal negation axiom)
- 9) The set of sequents which define the poset of the lattice of truth values $(X, \leq)$:
for any two $x, y \in X$, if $x \leq y$ then $x \vdash y \in \mathcal{G}$.

(INFERENCE RULES) $\mathcal{G}$ is closed under the following inference rules:

- 1) $\frac{\phi \vdash \psi, \psi \vdash \varphi}{\phi \vdash \varphi}$ (cut/ transitivity rule)
- 2) $\frac{\phi \vdash \psi, \phi \vdash \varphi}{\phi \vdash \psi \wedge \varphi}, \frac{\phi \vdash \psi, \varphi \vdash \psi}{\phi \vee \varphi \vdash \psi}$ (lower/upper lattice bound rules)
- 3) $\frac{\phi \vdash \psi}{\Box\phi \vdash \Box\psi}, \frac{\phi \vdash \psi}{\Diamond\phi \vdash \Diamond\psi}$ (monotonicity of modal operators rules)
- 4) $\frac{\phi \vdash \psi}{\neg\psi \vdash \neg\phi}$ (antitonicity of modal negation rule)
- 5) $\frac{\phi \vdash \psi}{\sigma(\phi) \vdash \sigma(\psi)}$ (substitution rule: σ is substitution (γ/p)).

Notice that the rules in point 2 are the consequences of diagonal mapping $\Delta : X \rightarrow Y$, where $Y = X \times X$ and $\Delta x = (x, x)$, (which is both additive and multiplicative modal operator), and its Galois adjunctions with the meet (multiplicative) and join (additive) operators $\wedge, \vee : Y \rightarrow X$, i.e., with $\Delta \dashv \wedge$ and $\vee \dashv \Delta$; that is $\Delta x \leq_Y (y, z)$ (i.e., $x \leq y$ and $x \leq z$) iff $x \leq \wedge(y, z) = y \wedge z$, and $x \vee y = \vee(x, y) \leq z$ iff $(x, y) \leq_Y \Delta z$ (i.e., $x \leq z$ and $y \leq z$).

The axioms from 1 to 5 and the rules 1 and 2 are taken from (Dunn, 1995) for the *DLL* and it was shown that this sequent based Gentzen-like system is sound and complete. If we omit the distributivity axiom 5, we obtain the system $\mathcal{G}$ for complete lattice logics. Thus, for a lattice based modal many-valued logics we obtain a *normal* modal Gentzen-like deductive system, where each sequent is a valid truth-preserving consequence pair defined by the poset of the complete lattice $(X, \leq)$ of truth values (which are also the constants of this modal propositional language $\mathcal{L}$), so that each occurrence of the symbol $\vdash$ can be substituted by the partial order $\leq$ of this complete lattice. $\square$

DEFINITION 2 (TRUTH-PRESERVING ENTAILMENT). — *For any two formulae $\phi, \psi \in \mathcal{L}$, the truth-preserving consequence pair (sequent), denoted by $\phi \vdash \psi$ is satisfied by a given valuation $v : \mathcal{L} \rightarrow X$ if $v(\phi) \leq v(\psi)$. This sequent is a tautology if it is satisfied by all valuations, i.e., when $\forall v \in \mathbb{V}_m. (v(\phi) \leq v(\psi))$.*

For a normal Gentzen-like sequent system $\mathcal{G}$ of the many-valued logic $\mathcal{L}$, with the set of sequents $\text{Seq}_{\mathcal{G}} \subseteq \mathcal{L} \times \mathcal{L}$ and a set of inference rules in $\text{Rul}_{\mathcal{G}}$, we tell that a many-valued valuation v is its model if it satisfies all sequents in $\mathcal{G}$. The set of all models of a given set of sequents (theory) Γ is denoted by $\text{Mod}_{\Gamma} =_{\text{def}} \{v \in \mathbb{V}_m \mid \forall (\phi \vdash \psi) \in \Gamma (v(\phi) \leq v(\psi))\} \subseteq \mathbb{V}_m \subset X^{\mathcal{L}}$.

PROPOSITION 3 (SOUNDNESS). — *All axioms of the Gentzen like sequent system $\mathcal{G}$, of a many-valued logic $\mathcal{L}$ based on complete lattice $(X, \leq)$ of algebraic truth values, are the tautologies, and all its rules are sound for model satisfiability and preserve the tautologies.*

PROOF. — It is straightforward to check (see the Example 1) that all axioms are tautologies (all constant sequents specify the poset of the complete lattice $(X, \leq)$, thus are tautologies). It is straightforward to check that all rules preserve the tautologies. Moreover, if all premises of any rule in $\mathcal{G}$ are satisfied by given many-valued valuation $v : \mathcal{L} \rightarrow X$, then also the deduced sequent of this rule is satisfied by the same valuation, *i.e.*, the rules are sound for the model satisfiability. ■

It is easy to verify that for any two $x, y \in X$ we have that $x \leq y$ iff $x \vdash y$, that is the truth-preserving entailment coincides with the partial truth-ordering in a lattice $(X, \leq)$. Notice that it is compatible with lattice operators, that is, for any two formulae $\phi, \psi \in \mathcal{L}$, $\phi \wedge \psi \vdash \psi$ and $\phi \vdash \psi \vee \phi$. This entailment imposes the following restrictions to the logic implication: in order to satisfy the Deduction Theorem " $z \vdash x \Rightarrow y$ iff $z \wedge x \vdash y$ " (*i.e.*, inference rules for elimination and introduction of the logic connective $\Rightarrow$, $\frac{z \vdash x \Rightarrow y}{z \wedge x \vdash y}$ and $\frac{z \wedge x \vdash y}{z \vdash x \Rightarrow y}$) by this entailment, the logic implication must satisfy (the case when $z = 1$) the requirement that for any $x, y \in X$, $x \Rightarrow y = 1$ iff $x \leq y$, while it must satisfy $x \wedge (x \Rightarrow y) \leq y$ in order to satisfy the Modus Ponens inference rule.

The particularity of this entailment is that any consequence pair (sequent) $\phi \vdash \psi$ is algebraically an equation $\phi \wedge \psi = \phi$ (or, $\phi \vee \psi = \psi$).

It is easy to verify, that in the case of the classic 2-valued propositional logic this entailment is equal to the classic propositional entailment, so that the truth-preserving entailment is only a generalization of the classic entailment for a many-valued propositional logics.

REMARK. — It is easy to observe that each sequent is, from the logic point of view, a 2-valued object so that all inference rules are embedded into the classic 2-valued framework, *i.e.*, given a bivaluation $\beta : \mathcal{L} \times \mathcal{L} \rightarrow \mathbf{2}$, we have that a sequent $s = \phi \vdash \psi$ is satisfied when $\beta(s) = 1$, so that we have the relationship between sequent bivaluations and many-valued valuations v used in Definition 2. □

From my point of view, this sequent feature, which is only an alternative formulation for the 2-valued classic logic, is *fundamental* in the framework of many-valued logics, where often the semantics for the entailment, based on algebraic matrices (X, D) is arbitrary: consider, for example fuzzy logic, where the subset of designated elements $D \subseteq X$ is an closed interval $[a, 1]$, where $0 < a \leq 1$ is an arbitrary value, so difficult to fix.

Thus, this correct definition of the 2-valued entailment in the sequent system $\mathcal{G}$, based only on the lattice ordering, can replace the current entailment based on the algebraic matrices (X, D) , where $D \subseteq X$ is the subset of designated elements, which is upward closed, that is, if $x \in D$ and $x \leq y$ then $y \in D$ (thus $1 \in D$), and where

the matrix-entailment, defined by $\phi \vdash_D \psi$, is valid iff $\forall v \in \mathbb{V}_m. (v(\phi) \in D \text{ implies } v(\psi) \in D)$. It is easy to verify also that $\phi \vdash \psi$ implies $\phi \vdash_D \psi$.

This opinion is based also on the consideration that all many-valued, *implication based* (IF), logic program languages, are based on clauses $A \leftarrow B_1 \wedge \dots \wedge B_n$ where A is a ground atom and B_i are ground literals (ground atoms or negation of ground atoms), we tell that it is *satisfied* by a valuation v iff $v(A) \leq v(B_1) \wedge \dots \wedge v(B_n)$; Thus, this clause is the sequent $(B_1 \wedge \dots \wedge B_n) \vdash A$, so that a logic program is in fact a set of sequents; the valuation v which satisfies all clauses (sequents) is a *model* for such a logic program, and demonstrates how we are able to define the models of many-valued logics without the necessity to define the algebraic matrices (X, D) .

The other class of "many valued" logic programs, Signed logic programs (Beckert *et al.*, 1999; Imaz *et al.*, 1994) and its subclass of *annotation based* (AB) programs (Blair *et al.*, 1989; Kifer *et al.*, 1992) a clause is of the form $A : f(\delta_1, \dots, \delta_n) \leftarrow B_1 : \delta_1, \dots, B_n : \delta_n$, which asserts "the certainty of the atom A is least (or is in) $f(\delta_1, \dots, \delta_n)$, whenever the certainty of the atom B_i is at least (or is in) δ_i , $1 \leq i \leq n$ ", where f is an n -ary computable function and δ_i is either a constant or a variable ranging over many-valued logic values. But they are not really many-valued logics, but a kind of "meta" many-valued logics, because all annotated literals $B_n : \delta_n$ are classic 2-valued objects (a valuation v is a model of the literal $B_n : \delta_n$ if $v(B_n) \geq \delta_n$ (or $v(B_n) \in \delta_n$)), and all connectives in the clauses are also classic 2-valued. In fact, in (Majkić, 2006e) it is shown that the annotated elements are 2-valued modal formulae, so that an annotated logic is a kind of 2-valued multi modal logics; the same result can be proven also in the more general case of Signed logic programming.

Thus we are able now to introduce the many-valued valuation-based (*i.e.*, model-theoretic) semantics for many-valued logics:

DEFINITION 4. — A many-valued model-theoretic semantics of a given many-valued logic $\mathcal{L}$, with a Gentzen system $\mathcal{G} = \langle \mathbb{L}, \vdash \rangle$, is the semantic deducibility relation $\models_m$, defined for any $\Gamma = \{s_i = (\phi_i \vdash \psi_i) \mid i \in I\}$ and sequent $s = (\phi \vdash \psi) \in \mathbb{L} \subseteq \mathcal{L} \times \mathcal{L}$ by: " $\Gamma \models_m s$ iff all many-valued models of Γ are the models of s ". That is: $\Gamma \models_m s$

iff $\forall v \in \mathbb{V}_m (\forall (\phi_i \vdash \psi_i) \in \Gamma (v(\phi_i) \leq v(\psi_i)) \text{ implies } v(\phi) \leq v(\psi))$

iff $\forall v \in \text{Mod}_\Gamma (\forall (\phi_i \vdash \psi_i) \in \Gamma (v(\phi_i) \leq v(\psi_i)) \text{ implies } v(\phi) \leq v(\psi))$

iff $\forall v \in \text{Mod}_\Gamma (v(\phi) \leq v(\psi))$.

It is easy to verify that any complete-lattice based many-valued logic has the Gentzen-like system $\mathcal{G} = \langle \mathbb{L}, \vdash \rangle$ (see the Example above) which is *normal* logic.

THEOREM 5. — The many-valued model theoretic semantics is an adequate semantics for a many-valued logic $\mathcal{L}$ specified by a Gentzen-like logic system $\mathcal{G} = \langle \mathbb{L}, \vdash \rangle$, that is, it is sound and complete. Consequently, $\Gamma \models_m s$ iff $\Gamma \vdash s$.

PROOF. — Let us prove that for any many valued model $v \in \text{Mod}_\Gamma$, the obtained sequent bivaluation $\beta = eq \circ < \pi_1, \wedge > \circ (v \times v) : \mathcal{L} \times \mathcal{L} \rightarrow \mathbf{2}$ is the characteristic

function of the closed theory $\Gamma_v = C(T)$ with $T = \{\phi \vdash x, x \vdash \phi \mid \phi \in \mathcal{L}, x = v(\phi)\}$.

From the definition of β we have that $\beta(\phi \vdash \psi) = \beta(\phi; \psi) = eq \circ < \pi_1, \wedge > \circ (v \times v)(\phi; \psi) = eq \circ < \pi_1, \wedge > (v(\phi), v(\psi)) = eq < \pi_1(v(\phi), v(\psi)), \wedge(v(\phi), v(\psi)) > = eq(v(\phi), \wedge(v(\phi), v(\psi))) = eq(v(\phi), v(\phi) \wedge v(\psi))$, where $\pi_1 : X \times X \rightarrow X$ is the first projection, $eq : X \times X \rightarrow \mathbf{2} \subseteq X$ is the equality characteristic function such that $eq(x, y) = 1$ if $x = y$.

Thus $\beta(\phi \vdash \psi) = 1$ iff $v(\phi) \leq v(\psi)$, i.e., when this sequent is satisfied by v .

1) Let us show that for any sequent s , $s \in \Gamma_v$ implies $\beta(s) = 1$:

First of all, any sequent $s \in T$ is of the form $\varphi \vdash x$ or $x \vdash \varphi$, where $x = v(\varphi)$, so that it is satisfied by v (holds that $v(\varphi) \leq v(\varphi)$ in both cases). Consequently, *all* sequents in T are satisfied by v .

By the Proposition 3 we have that all inference rules in $\mathcal{G}$ are sound w.r.t the model satisfiability, thus for any deduction $T \Vdash s$ (i.e., $s \in \Gamma_v$) where all sequents in premises are satisfied by the many-valued valuation (model) v , also the deduced sequent $s = (\phi \vdash \psi)$ must be satisfied, that is must hold $v(\phi) \leq v(\psi)$, i.e., $\beta(s) = 1$.

2) Let us show that for any sequent s , $\beta(s) = 1$ implies $s \in \Gamma_v$:

For *any* sequent $s = (\phi \vdash \psi) \in \mathcal{L} \times \mathcal{L}$ if $\beta(s) = 1$ then $x = v(\phi) \leq v(\psi) = y$ (i.e., s is satisfied by v). From the definition of T , we have that $\phi \vdash x, y \vdash \psi \in T$, and from $x \leq y$ we have $x \vdash y \in Ax_{\mathcal{G}}$ (where $Ax_{\mathcal{G}}$ are axioms (sequents) in $\mathcal{G}$, with $\{x \vdash y \mid x, y \in X, x \leq y\} \subseteq Ax_{\mathcal{G}}$, thus satisfied by every valuation) by the transitivity rule, from $\phi \vdash x, x \vdash y, y \vdash \psi$, we obtain that $T \Vdash (\phi \vdash \psi)$, i.e., $s = (\phi \vdash \psi) \in C(T) = \Gamma_v$.

So, from 1) and 2) we obtain that $\beta(s) = 1$ iff $s \in \Gamma_v$, i.e., the sequent bivaluation β is the characteristic function of a closed set. Consequently, any many-valued *model* v of this many-valued logic $\mathcal{L}$ corresponds to the *closed* bivaluation β which is a characteristic function of a closed theory of sequents: we define the set of all closed bivaluations obtained from the set of many-valued models $v \in Mod_{\Gamma}$: $Biv_{\Gamma} = \{\Gamma_v \mid v \in Mod_{\Gamma}\}$. From the fact that Γ is satisfied by every $v \in Mod_{\Gamma}$ we have that for every $\Gamma_v \in Biv_{\Gamma}$, $\Gamma \subseteq \Gamma_v$, so that $C(\Gamma) = \bigcap Biv_{\Gamma}$ (intersection of closed sets is a closed set also). Thus, for $s = (\phi \vdash \psi)$, $\Gamma \models_m s$

iff $\forall v \in Mod_{\Gamma} (\forall(\phi_i \vdash \psi_i) \in \Gamma (v(\phi_i) \leq v(\psi_i)))$ implies $v(\phi) \leq v(\psi)$

iff $\forall v \in Mod_{\Gamma} (\forall(\phi_i \vdash \psi_i) \in \Gamma (\beta(\phi_i \vdash \psi_i) = 1))$ implies $\beta(\phi \vdash \psi) = 1$

iff $\forall v \in Mod_{\Gamma} (\forall(\phi_i \vdash \psi_i) \in \Gamma ((\phi_i \vdash \psi_i) \in \Gamma_v))$ implies $s \in \Gamma_v$

iff $\forall \Gamma_v \in Biv_{\Gamma} (\Gamma \subseteq \Gamma_v$ implies $s \in \Gamma_v)$

iff $\forall \Gamma_v \in Biv_{\Gamma} (s \in \Gamma_v)$, because $\Gamma \subseteq \Gamma_v$ for each $\Gamma_v \in Biv_{\Gamma}$

iff $s \in \bigcap Biv_\Gamma = C(\Gamma)$, that is,

iff $\Gamma \Vdash s$. ■

Thus, in order to define the model-theoretic semantics for a many-valued logics we do not need to define the problematic matrices: we are able to use only the many-valued valuations, and *many-valued models* (i.e., valuations which satisfy all sequents in Γ of a given many-valued logic $\mathcal{L}$). This point of view is used also for definition of a new Representation Theorem for many-valued logics in (Majkić, 2006d).

Differently from the classic logic where a formula is a theorem if it is true in all models of the logic, here, in a many-valued logic $\mathcal{L}$, but specified by a set of sequents in Γ , for a formula $\phi \in \mathcal{L}$ which has the same value $x \in X$ (for any algebraic truth-value x) for all many-valued models $I \in Mod_\Gamma$, we have that its sequent-based version $\phi \vdash x$ and $x \vdash \phi$ are theorems; that is $\forall I \in Mod_\Gamma (I(\phi) = x) \text{ iff } (\Gamma \Vdash (\phi \vdash x) \text{ and } \Gamma \Vdash (x \vdash \phi))$.

But such a value $x \in A$ does not need to be a designated element $x \in D$, as in matrix semantics for a many-valued logic, and it explains why we do not need the rigid semantic specification by matrix designated elements. Thus, by a translation of a many-valued logic $\mathcal{L}$ into its "meta" sequent-based 2-valued logic, we obtain an unambiguous theory of inference without the introduction of problematic matrices.

REMARK. — There are also two other ways, alternative to 2-valued sequent systems, to reduce the many-valued logics into "meta" 2-valued logics: The first is based on the Ontological encapsulation (Majkić, 2004; Majkić, 2006e), where each many-valued proposition (or many-valued ground atom $p(a_1, \dots, a_n)$) is ontologically encapsulated into the "flattened" 2-valued atom $p_F(a_1, \dots, a_n, x)$ (by enlarging original atoms with new logic variable whose domain of values is the set of algebraic truth-values of a complete lattice $x \in X$: ruffly, " $p(a_1, \dots, a_n)$ has a value x " iff $p_F(a_1, \dots, a_n, x)$ is true). In fact such an atom is equivalent to the following formula of sequents $(p(a_1, \dots, a_n) \vdash x) \wedge (x \vdash p(a_1, \dots, a_n))$.

The other way, which can be used in the case when a complete lattice $(X, \leq)$ of algebraic truth-values is *finite*, is to represent this sequent formula by a modal 2-valued atom $\Box_x p(a_1, \dots, a_n)$ (where $\Box_x$ is an universal modal operator for a given algebraic truth-value in X) with autoreferential semantics (Majkić, 2006e), where the set of possible worlds of this multi modal logic is the set of algebraic truth values in the lattice $(X, \leq)$. □

Based on this Genzen-like sequent deductive system $\mathcal{G}$ we are able to define the equivalence relation $\approx_L$ between the formulae of any modal propositional logic based on a complete lattice in order to define the Lindenbaum algebra for this logic, $(\mathcal{L}/\approx_L, \sqsubseteq)$, where for any two formulae $\phi, \psi \in \mathcal{L}$,

- a) $\phi \approx_L \psi$ iff $\phi \vdash \psi$ and $\psi \vdash \phi$, i.e.,
iff $\forall v. (v(\phi) \leq v(\psi))$ and $\forall v. (v(\phi) \geq v(\psi))$.

Thus the quotient algebra $\mathcal{L}/\approx_L$ has as elements equivalence classes, denoted by $[\phi]$, and the partial ordering $\sqsubseteq$, defined by

b) $[\phi] \sqsubseteq [\psi]$ iff $\phi \vdash \psi$ (i.e., if $\phi \leq \psi$).

It is easy to verify that each equivalence class (set of all equivalent formulae w.r.t $\approx_L$) $[\phi]$ has exactly one constant $x \in X$ which is element of this equivalence class, and we can use it as the representation element for this equivalence class: so that every formula in this equivalence class has the same truth-value as this constant.

Thus, we have the bijection $is : \mathcal{L} / \approx_L \rightarrow X$ between elements in the complete lattice $(X, \leq)$ and elements in the Lindenbaum algebra, such that for any equivalence class $[\phi] \in \mathcal{L} / \approx_L$, the constant $is([\phi]) \in X$ is the representation element for this equivalence class. It is easy to extend this bijection into an isomorphism between the original algebra and this Lindenbaum algebra, by definition of correspondent connectives in the Lindenbaum algebra, for example:

$$[\phi] \wedge_L [\psi] =_{def} is^{-1}(is([\phi]) \wedge is([\psi])),$$

$$[\phi] \vee_L [\psi] =_{def} is^{-1}(is([\phi]) \vee is([\psi])),$$

$$\neg_L [\phi] =_{def} is^{-1}(\neg(is([\phi]))), \text{ etc.}$$

In the autoreferential semantics we will assume that each equivalence class of formulae $[\phi]$ in this Lindenbaum algebra corresponds to one “state-description”, that is to one possible world in the Kripke-style semantics for the original many-valued modal logic. But, from the isomorphism is we can take, instead of equivalence class $[\phi]$, only its representation element $x = is([\phi]) \in X$.

Consequently the set of possible worlds in this autoreferential semantics corresponds to the set of truth values in the complete lattice $(X, \leq)$.

3. Autoreferential representation theorem for complete lattices

Generally lattices arise concretely as the substructures of closure systems (intersection systems) where a closure system is a family $\mathcal{F}(X)$ of subsets of a set X such that $X \in \mathcal{F}(X)$ and if $A_i \in \mathcal{F}(X), i \in I$, then $\bigcap_{i \in I} A_i \in \mathcal{F}(X)$. Then the representation problem for general lattices is to establish that every lattice can be viewed, up to an isomorphism, as a collection of subsets in a closure system (on some set X), closed under the operations of the system.

Closure operators Γ are canonically obtained by the composition of two maps of Galois connections. The Galois connections can be obtained from any binary relation on a set X (Birkhoff, 1940) (Birkhoff *polarity*) in a canonical way: If (X, R) is a set with a particular relation on a set X , $R \subseteq X \times X$, with mappings $\lambda : \mathcal{P}(X) \rightarrow \mathcal{P}(X)^{OP}, \rho : \mathcal{P}(X)^{OP} \rightarrow \mathcal{P}(X)$, such that for any $U, V \in \mathcal{P}(X)$, $\lambda U = \{x \in X \mid \forall u \in U. ((u, x) \in R)\}$, $\rho V = \{x \in X \mid \forall v \in V. ((x, v) \in R)\}$, where $\mathcal{P}(X)$ is the powerset poset with bottom element empty set $\emptyset$ and top element X , and $\mathcal{P}(X)^{OP}$ its dual (with $\subseteq^{OP}$ inverse of $\subseteq$), then we obtain the induced Galois connection $\lambda \dashv \rho$, i.e., $\lambda U \subseteq^{OP} V$ iff $U \subseteq \rho V$.

The following lemma is useful for the relationship of these set-based operators with the operation of negation in the complete lattices.

LEMMA 6 (INCOMPATIBILITY RELATION). — *Let $(X, \leq)$ be a complete lattice. Then we can use a binary relation $R \subseteq X \times X$ as an incompatibility relation for set-based negation operators λ and ρ , with the following properties: for any $U, V \subseteq X$,*

- 1) $\lambda(U \cup V) = \lambda U \bigcup^{OP} \lambda V = \lambda U \cap \lambda V$ with $\lambda \emptyset = \emptyset^{OP} = X$ (additivity)
- 2) $\rho(U \bigcap^{OP} V) = \rho(U \cup V) = \rho U \cap \rho V$ with $\rho X^{OP} = \rho \emptyset = X$ (multiplicativity)
- 3) while $\lambda(U \cap V) \geq \lambda U \cup \lambda V$, $\rho(U \cup V) \geq \rho U \bigcup \rho V$ and $\lambda \rho V \geq V$, $\rho \lambda U \geq U$.

We will consider the case when (X, R) is a complete lattice with the binary relation R equal to the partial order $\leq$ in the lattice X . The resulting Galois connection on this partial order is the familiar Dedekind-McNeile Galois connection of antitonic mappings λ and ρ (We denote by $\downarrow x$ the ideal $\{y \in X \mid y \leq x\}$):

$$\lambda U = \{x \in X \mid \forall u \in U. (u \leq x)\},$$

$$\rho V = \{x \in X \mid \forall v \in V. (x \leq v)\}.$$

Setting $\Gamma = \rho \lambda : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$, the operator Γ is a monotone mapping and a closure operator, such that for any $U, V \in \mathcal{P}(X)$

- 1) $U \subseteq V$ implies $\Gamma(U) \subseteq \Gamma(V)$,
- 2) $U \subseteq \Gamma(U)$, for any $U \in \mathcal{P}(X)$,
- 3) $\Gamma \Gamma(U) = \Gamma(U)$

The set $\{U \in \mathcal{P}(X) \mid U = \Gamma(U)\}$ is called the set of *stable* (closed) sets.

It is easy to verify that for every subset $S \subseteq X$, we obtain $\Gamma(S) = \downarrow \bigvee(S)$, which is a multiplicative monotone operator: the closure operator Γ satisfies the multiplicative property $\Gamma(U \cap V) = \Gamma(U) \cap \Gamma(V)$, and $\Gamma(X) = X$ which is top element in $\mathcal{P}(X)$.

Thus, $\Gamma = \downarrow \bigvee$ is a *universal modal operator* for the complete lattice $(\mathcal{P}(X), \subseteq)$ (which is a Boolean algebra, that is, a *complemented distributive* lattice), so that we obtain the modal algebra $(\mathcal{P}(X), \subseteq, \bigcap, \bigcup, \Gamma)$ for the complete lattice $(X, \leq)$.

Notice that $\Gamma(\emptyset) = \Gamma(\{0\}) = \{0\}$. Thus, from the monotone property of Γ and from $\emptyset \subseteq U$ (for any $U \in \mathcal{P}(X)$) we obtain that $\{0\} = \Gamma(\emptyset) \subseteq \Gamma(U)$. That is, any stable (closed) set contains the bottom element $0 \in X$, and that the minimal stable set is $\{0\}$ and not an empty set $\emptyset$, while, naturally, the maximal stable set is X .

Moreover, each stable set is a downclosed ideal and there is the bijection between the set of stable sets and the set of algebraic truth-values in X , so that we are able to define the representation theorem for a complete lattice $(X, \leq)$ as follows:

PROPOSITION 7 (AUTOREFERENTIAL REPRESENTATION THEOREM FOR COMPLETE LATTICES). — *Let $(X, \leq)$ be a complete lattice. We define the Dedekind-McNeile coalgebra $\downarrow : X \rightarrow \mathcal{P}(X)$, where $\downarrow = \Gamma \text{in}$, $\Gamma = \rho \lambda$ is a Dedekind-McNeile closure operator and $\text{in} : X \rightarrow \mathcal{P}(X)$ the inclusion map $x \mapsto \{x\}$, such that for any two $x, y \in X$ holds (we denote by $-$ the set subtraction)*

$$1) \quad \bigcup \{S \in \mathcal{P}(X) \mid S \cap \downarrow x \subseteq \downarrow y\} = (X - \downarrow x) \cup \downarrow y.$$

We denote by $\mathcal{F}(X) = \{\downarrow x \mid x \in X\}$ the closure system because for each $x \in X$, $\downarrow x$ is a stable set. Then, the $(\mathcal{F}(X), \subseteq)$ is a complete lattice with meet operator $\cap$ (set intersection), and join operator $\uplus = \Gamma \bigcup = \downarrow \bigvee \bigcup$ different from the set union $\bigcup$, such that the AUTOREFERENTIAL REPRESENTATION ISOMORPHISM holds:

$$2) \quad \downarrow : (X, \leq, \wedge, \vee, 0, 1) \simeq (\mathcal{F}(X), \subseteq, \cap, \uplus, \{0\}, X).$$

PROOF. — We will show that each stable set $U \in \mathcal{F}(X)$ is an ideal in $(X, \leq)$. In fact it holds that for any $x \in X$, $\downarrow x = \Gamma \text{in}(x) = \rho\lambda(\{x\}) = \{x' \in X \mid x' \leq x\}$ is an ideal. Thus, for any two $x, y \in X$ we have that

- 1) If $x \leq y$ then $\downarrow x \subseteq \downarrow y$,
- 2) $\downarrow x \cap \downarrow y = \{x' \in X \mid x' \leq x\} \cap \{y' \in X \mid y' \leq y\} = \{z \in X \mid z \leq x \wedge y\} = \downarrow (x \wedge y) \in \mathcal{F}(X)$,
- 3) $\downarrow x \uplus \downarrow y = \downarrow \bigvee \bigcup (\downarrow x, \downarrow y) = \downarrow \bigvee (\downarrow x \cup \downarrow y) = \downarrow \bigvee (\{x' \in X \mid x' \leq x\} \cup \{y' \in X \mid y' \leq y\}) = \downarrow (x \vee y) \in \mathcal{F}(X)$.

The operator $\uplus$ is the join operator in $\mathcal{F}(X)$, that is holds $\downarrow x \subseteq \downarrow y$ iff $\downarrow x \uplus \downarrow y = \downarrow y$.

Thus the original lattice $(X, \leq)$ is isomorphic to the lattice $(\mathcal{F}(X), \subseteq)$ via map $x \mapsto \downarrow x$. It is easy to verify that the inverse of $\downarrow$, i.e., $\downarrow^{-1} = (\Gamma \text{in})^{-1} = \text{in}^{-1} \Gamma^{-1} : \mathcal{P}(X) \rightarrow X$ is equal to supremum $\bigvee$, that is $\downarrow^{-1}(U) = \bigvee \{x \in U\}$, with $\bigvee \downarrow x = x$, that is, $\bigvee \downarrow = \text{id}_X$ and $\downarrow \bigvee = \text{id}_{\mathcal{F}(X)}$ are the identity functions for x and $\mathcal{F}(X)$ respectively. It is easy to verify that $\mathcal{F}(X)$ is a complete lattice with bottom element $\downarrow 0 = \{0\}$ and top element $\downarrow 1 = X$, and for any $U = \downarrow x, V = \downarrow y \in \mathcal{F}(X)$, $U \cap V = \downarrow (x \wedge y) \in \mathcal{F}(X)$ and $U \uplus V = \downarrow (x \vee y) \in \mathcal{F}(X)$. ■

REMARK. — What is important to notice is that the autoreferential representation, based only on the lattice ordering of truth values, naturally introduce the modality in this logic. In fact, the universal modal operator $\Gamma : \mathcal{P}(X) \rightarrow \mathcal{F}(X) \subset \mathcal{P}(X)$, based on the partial order of the complete lattice $(X, \leq)$, plays the fundamental role for the modal disjunction $\uplus$ in the canonical autoreferential algebra $\mathcal{F}(X)$. We need it because generally for any two $x, y \in X$ we have that $\downarrow x \cup \downarrow y \notin \mathcal{F}(X)$. Its restriction on $\mathcal{F}(X)$ is an *identity* function $\text{id}_{\mathcal{F}(X)}$, which is a selfadjoint (universal and existential) modal operator. If we denote by $\Box$ the *logic* modal operator correspondent to the algebraic modal operator Γ , we can define the join operator $\vee$ (many-valued disjunction) from the *classic* disjunction $\vee_c$ (for which holds $\downarrow \vee_c = \bigcup \downarrow$, that is, $\downarrow (x \vee_c y) = \downarrow x \cup \downarrow y$) and this modal operator, as follows: for any two logic formulae ϕ and ψ : $\phi \vee \psi = \Box(\phi \vee_c \psi)$. This property will be used when we will define the Kripke style semantics for this many-valued logic.

Notice also that in Lewis's approach he distinguishes the standard or *extensional* disjunction $\vee_c$, from the *intensional* disjunction $\vee$ "in such that at least one of the disjoined propositions is "necessarily" true" (Lewis, 1912) (1912, p.523). In the same way he defined also the "strict" logic implication $\phi \Rightarrow \psi = \Box(\phi \Rightarrow_c \psi)$ where $\Rightarrow_c$ is the classic extensional implication such that $\phi \Rightarrow_c \psi = \neg_c \phi \vee_c \psi$ (here $\neg_c$ denotes

the classic logic negation). In what follows we will see that this property holds also for a many-valued implication defined as relative pseudo-complement in a complete distributive lattice X , that is for the *intuitionistic* logic implication.

Thus, the autoreferential semantics for many-valued logic, based on the partial order of the lattice, defines the intensional disjunction and implication as a many-valued generalization of Lewis's 2-valued logic.

It remains to explain what an *algebraic* universal modal operator for a lattice X is, corresponding to the universal algebraic modal operator Γ in $\mathcal{P}(X)$. The answer is simple: it is an identity operator $id : X \rightarrow X$. In fact we have that the homomorphism $\downarrow$ between the many-valued algebra over X and the powerset algebra over $\mathcal{P}(X)$ is extended by $\downarrow id = \Gamma \downarrow$. Then we obtain that $\downarrow x = \downarrow id(x) = \Gamma \downarrow x = \downarrow \bigvee \downarrow x = \downarrow x$, and that $\downarrow (x \vee y) = \downarrow (id \vee_c (x, y)) = (\text{from } \downarrow id = \Gamma \downarrow) = \Gamma \downarrow \vee_c(x, y) = (\text{from } \downarrow \vee_c = \bigcup \downarrow) = \Gamma \bigcup \downarrow (x, y) = \Gamma \bigcup (\downarrow x, \downarrow y) = \Gamma(\downarrow x \bigcup \downarrow y)$.

While, $\downarrow (x \wedge y) = \downarrow (id \wedge_c (x, y)) = (\text{from } \downarrow id = \Gamma \downarrow) = \Gamma \downarrow \wedge_c(x, y) = (\text{from } \downarrow \wedge_c = \bigcap \downarrow) = \Gamma \bigcap \downarrow (x, y) = \Gamma \bigcap (\downarrow x, \downarrow y) = \Gamma(\downarrow x \bigcap \downarrow y) = (\text{the operator } \Gamma \text{ is an identity for stable sets}) = \downarrow x \bigcap \downarrow y = \downarrow (x \wedge y)$.

That is, the modal operator id has no effect on the conjunction, because $\downarrow x \bigcap \downarrow y \in \mathcal{F}(X)$ is a stable set, different from $\downarrow x \bigcup \downarrow y \notin \mathcal{F}(X)$ which is not.

Thus we obtain the following modal version of the autoreferential representation for many-valued algebras based on a complete lattice X :

$$\downarrow : (X, \leq, \wedge, \vee, id, 0, 1) \simeq (\mathcal{F}(X), \subseteq, \bigcap, \Gamma \bigcup, id_{\mathcal{F}(X)}, \{0\}, X) \\ \not\simeq (\mathcal{P}(X), \subseteq, \bigcap, \bigcup, \Gamma, \emptyset, X),$$

i.e., $\mathcal{F}(X)$ is not a subalgebra of the powerset Boolean distributive algebra $\mathcal{P}(X)$ with a modal operator Γ , as in the standard representation theorems for algebras. In fact, here we do not impose that X (or $\mathcal{F}(X)$) has to be distributive, as it holds for any Boolean sublattice. $\square$

EXAMPLE 8. — The smallest *nontrivial* bilattice is Belnap's 4-valued bilattice (Belnap, 1977; Fitting, 1991; Majkić, 2007) $\mathcal{B} = \{t, f, \perp, \top\}$ where t is *true*, f is *false*, $\top$ is inconsistent (both true and false) or *possible*, and $\perp$ is *unknown*. As Belnap observed, these values can be given two natural orders: *truth* order, $\leq_t$, and *knowledge* order, $\leq_k$, such that $f \leq_t \top \leq_t t$, $f \leq_t \perp \leq_t t$, $\perp \bowtie_t \top$ and $\perp \leq_k f \leq_k \top$, $\perp \leq_k t \leq_k \top$, $f \bowtie_k t$. That is, the bottom element 0 for $\leq_t$ ordering is f , and for $\leq_k$ ordering is $\perp$, and the top element 1 for $\leq_t$ ordering is t , and for $\leq_k$ ordering is $\top$.

Meet and join operators under $\leq_t$ are denoted $\wedge$ and $\vee$; they are natural generalizations of the usual conjunction and disjunction notions. Meet and join under $\leq_k$ are denoted $\otimes$ and $\oplus$, such that hold: $f \otimes t = \perp$, $f \oplus t = \top$, $\top \wedge \perp = f$ and $\top \vee \perp = t$.

The *bilattice negation* (Ginsberg, 1988) is given by $\neg f = t$, $\neg t = f$, $\neg \perp = \perp$ and $\neg \top = \top$. In what follows we will use the *relative pseudo-complements* for implication (Belnap's 4-valued lattice is distributive), defined by $x \multimap y = \bigvee \{z \mid z \wedge x \leq_t y\}$, and the *pseudo-complements* for negation, defined by $\neg_t x = x \multimap f$ (which is different from the bilattice negation $\neg$). It is easy to see that holds the De Morgan

law $\neg_t(x \wedge y) = \neg_t x \vee \neg_t y$. We have that, w.r.t the truth ordering $\leq_t$, $\downarrow f = \{f\}$, $\downarrow \perp = \{f, \perp\}$, $\downarrow \top = \{f, \top\}$ and $\downarrow t = \mathcal{B}$. Thus, $\downarrow \perp \uplus \downarrow \top = \downarrow \bigvee (\downarrow \perp \cup \downarrow \top) = \downarrow \bigvee \{f, \perp, \top\} = \downarrow t = \mathcal{B} \neq \downarrow \perp \cup \downarrow \top$, that is $\uplus \neq \cup$. $\square$

It is easy to verify that the join operator $\uplus$ in the compact set-based representation $\mathcal{F}(X)$ reduces to the standard set union $\cup$ when the complete lattice $(X, \leq)$ is a total ordering (as, for example, the fuzzy logic with the closed interval of reals $X = [0, 1]$ for the set of truth values). It can be extended to all *distributive* lattices, such that for any $x \in X$ and $Y \subseteq X$, $x \wedge (\bigvee Y) = \bigvee \{x \wedge y \mid y \in Y\}$.

From Birkhoff's representation theorem (Birkhoff, 1940) for distributive lattices, every finite (thus complete) distributive lattice is isomorphic to the lattice of lower sets of the poset of join-irreducible elements.

PROPOSITION 9 (0-LIFTED BIRKHOFF ISOMORPHISM, (BIRKHOFF, 1940)). — *Let X be a complete distributive lattice, then we define the following mapping $\downarrow^+ : X \rightarrow \mathcal{P}(X)$: for any $x \in X$, $\downarrow^+ x = \downarrow x \cap \hat{X}$, where $\hat{X} = \{y \mid y \in X \text{ and } y \text{ is join-irreducible}\} \cup \{0\}$.*

We define the set $X^+ = \{\downarrow^+ a \mid a \in X\} \subseteq \mathcal{P}(X)$, so that $\downarrow^+ \bigvee = id_{X^+} : X^+ \rightarrow X^+$ and $\bigvee \downarrow^+ = id_X : X \rightarrow X$. Thus, the operator $\downarrow^+$ is the inverse of the supremum operation $\bigvee : X^+ \rightarrow X$. The set $(X^+, \subseteq)$ is a complete lattice, such that there is the following 0-lifted Birkhoff isomorphism $\downarrow^+ : (X, \leq, \wedge, \vee) \simeq (X^+, \subseteq, \cap, \cup)$.

PROOF. — Let us show the homomorphic property of $\downarrow^+$:

$$\begin{aligned} \downarrow^+(x \wedge y) &= \downarrow(x \wedge y) \cap \hat{X} = (\downarrow x \cap \downarrow y) \cap \hat{X} = (\downarrow x \cap \hat{X}) \cap (\downarrow y \cap \hat{X}) = \\ &= \downarrow^+ x \cap \downarrow^+ y, \text{ and} \\ \downarrow^+(x \vee y) &= \downarrow(x \vee y) \cap \hat{X} = (\downarrow x \cup \downarrow y) \cap \hat{X} = (\downarrow x \cap \hat{X}) \cup (\downarrow y \cap \hat{X}) = \\ &= \downarrow^+ x \cup \downarrow^+ y. \end{aligned}$$

The isomorphic property holds from Birkhoff's theorem. $\blacksquare$

The name “lifted” is used to denote the difference from the original Birkhoff's isomorphism: that is, we have that for any $x \in X$, $0 \in \downarrow^+ x$, so that $\downarrow^+ x$ is never an empty set (it is lifted by the bottom element 0).

Notice that when X is a distributive lattice then $(X^+, \subseteq, \cap, \cup)$ is a *subalgebra* of the powerset Boolean algebra $(\mathcal{P}(X), \subseteq, \cap, \cup)$, differently from the case when X is not distributive. Thus, we have

$$\begin{aligned} \downarrow^+ : (X, \leq, \wedge, \vee, id_X) &\simeq (X^+, \subseteq, \cap, \cup, id_{X^+}) \subseteq (\mathcal{P}(X), \subseteq, \cap, \cup, \Gamma^+), \\ \text{where } \Gamma^+ &= \downarrow^+ \bigvee : \mathcal{P}(X) \rightarrow \mathcal{P}(X) \text{ is a modal algebraic monotone multiplicative} \\ &\text{operator. Its reduction to the subset of join-irreducible elements } \hat{X} \subseteq X \text{ is equal to} \\ &\Gamma^+ = \Gamma : \mathcal{P}(\hat{X}) \rightarrow \mathcal{P}(\hat{X}). \end{aligned}$$

EXAMPLE 10. — Belnap's bilattice in Example 1, is a distributive lattice w.r.t the $\leq_t$ ordering with two join-irreducible elements $\perp$ and $\top$. In that case we have that $\downarrow^+ t = \downarrow^+ (\perp \vee \top) = \downarrow^+ \perp \cup \downarrow^+ \top = \downarrow \perp \cup \downarrow \top = \{f, \perp, \top\} \neq \downarrow t = \mathcal{B}$. $\square$

Now we will introduce a hierarchy of negation operators for complete lattices, based on their homomorphic properties: the negation with the lowest requirements (such that it inverts the truth ordering of the lattice of truth values and is able to pro-

duce falsity and truth, *i.e.*, the bottom and top elements of the lattice), denominated "general" negation, can be defined in any complete lattice (see the example below).

DEFINITION 11 (HIERARCHY OF NEGATION OPERATORS). — *Let $(X, \leq, \wedge, \vee)$ be a complete lattice. Then we define the following hierarchy of negation operators on it:*

- 1) A **general negation** is a monotone mapping between posets ($\leq^{OP}$ is inverse of $\leq$), $\neg : (X, \leq) \rightarrow (X, \leq)^{OP}$, such that $\{0, 1\} \subseteq \{y = \neg x \mid x \in X\}$.
- 2) A **split negation** is a general negation extended into the join-semilattice homomorphism, $\neg : (X, \leq, \vee) \rightarrow (X, \leq, \vee)^{OP}$, with $(X, \leq, \vee)^{OP} = (X, \leq^{OP}, \vee^{OP})$, $\vee^{OP} = \wedge$.
- 3) A **constructive negation** is a general negation extended into full lattice homomorphism, $\neg : (X, \leq, \wedge, \vee) \rightarrow (X, \leq, \wedge, \vee)^{OP}$, with $(X, \leq, \wedge, \vee)^{OP} = (X, \leq^{OP}, \wedge^{OP}, \vee^{OP})$, and $\wedge^{OP} = \vee$.
- 4) A **De Morgan negation** is a constructive negation when the lattice homomorphism is an involution ($\neg\neg x = x$).

The names given to these different kinds of negations follow from the fact that a split negation introduces the second right adjoint negation, a constructive negation satisfies the constructive requirement (as in Heyting algebras) $\neg\neg x \geq x$, while a De Morgan negation satisfies well known De Morgan laws:

LEMMA 12 (NEGATION PROPERTIES). — *Let $(X, \leq)$ be a complete lattice. Then the following properties for negation operators hold: for any $x, y \in X$,*

- 1) *for general negation:* $\neg(x \vee y) \leq \neg x \wedge \neg y$, $\neg(x \wedge y) \geq \neg x \vee \neg y$, with $\neg 0 = 1$, $\neg 1 = 0$.
- 2) *for split negation:* $\neg(x \vee y) = \neg x \wedge \neg y$, $\neg(x \wedge y) = \neg x \vee \neg y$. *It is an additive modal operator with right adjoint (multiplicative) negation $\sim : (X, \leq)^{OP} \rightarrow (X, \leq)$, and the Galois connection $\neg x \leq^{OP} y$ iff $x \leq \sim y$, such that $\sim \neg x \geq x$ and $\neg \sim x \geq x$.*
- 3) *for constructive negation:* $\neg(x \vee y) = \neg x \wedge \neg y$, $\neg(x \wedge y) = \neg x \vee \neg y$. *It is a selfadjoint operator, $\neg = \sim$, with $\neg\neg x \geq x$ satisfying the proto De Morgan inequalities $\neg(\neg x \vee \neg y) \geq x \wedge y$ and $\neg(\neg x \wedge \neg y) \geq x \vee y$.*
- 4) *for De Morgan negation ($\neg\neg x = x$): it satisfies also De Morgan laws $\neg(\neg x \vee \neg y) = x \wedge y$ and $\neg(\neg x \wedge \neg y) = x \vee y$, and is contrapositive, *i.e.*, $x \leq y$ iff $\neg x \geq \neg y$.*

PROOF. —

- 1) From the definition of a *general negation* as *monotonic* mapping between posets, $\neg : (X, \leq) \rightarrow (X, \leq)^{OP}$, we have that $x \leq y$ implies $\neg x \leq^{OP} \neg y$, *i.e.* it inverts the ordering, $\neg x \geq \neg y$. Thus, $x \wedge y \leq x$ and $x \wedge y \leq y$ implies $\neg(x \wedge y) \geq \neg x$ and $\neg(x \wedge y) \geq \neg y$, consequently $\neg(x \wedge y) \geq \neg x \vee \neg y$. Analogously, $x \vee y \geq x$ and $x \vee y \geq y$ implies $\neg(x \vee y) \leq \neg x$ and $\neg(x \vee y) \leq \neg y$, consequently $\neg(x \vee y) \leq \neg x \wedge \neg y$. From the property $\forall y (0 \leq y \text{ implies } \neg 0 \geq \neg y)$ and the fact that $1 \in \{z = \neg y \mid y \in X\}$ must hold $\neg 0 \geq 1$, *i.e.*, $\neg 0 = 1$. Analogously we can show also that $\neg 1 = 0$.

2) From the homomorphic definition of the *split* negation (which is also general, thus with $\neg(x \wedge y) \geq \neg x \vee \neg y$) we have that $\neg(x \vee y) = \neg x \vee^{OP} \neg y = \neg x \wedge \neg y$, and from $\neg 0 = 1 = 0^{OP}$ we conclude that it is a monotone additive mapping, thus it has the right multiplicative adjoint $\sim: (X, \leq)^{OP} \rightarrow (X, \leq)$ with the Galois connection $\neg \dashv \sim$, i.e., $\neg x \leq^{OP} y$ iff $x \leq \sim y$, that is, $\neg x \geq y$ iff $x \leq \sim y$. The operators $\neg \sim$ and $\sim \neg$ are the closure operators in $\mathcal{P}(X)$, it thus holds that $\neg \sim x \geq x$ and $\sim \neg x \geq x$.

3) The *constructive* negation is also a split negation, thus it is an additive modal operator. Let us show that it is also multiplicative, consequently self adjoint, i.e., $\sim = \neg$: it holds from the lattice homomorphism for meet operators, $\neg(x \wedge y) = \neg x \wedge^{OP} \neg y$ (i.e., $= \neg x \vee \neg y$), and from $\neg 1 = 0 = 1^{OP}$. Thus $\neg \neg x = \neg \sim x \geq x$ is its constructive property. From this constructive property and the additive and multiplicative properties we obtain that $\neg(\neg x \wedge \neg y) = \neg \neg x \vee \neg \neg y \geq x \vee y$ (because $\vee$ is monotone for both arguments). Analogously, $\neg(\neg x \vee \neg y) = \neg \neg x \wedge \neg \neg y \geq x \wedge y$ (because also $\wedge$ is monotone for both arguments), i.e., we obtained the proto De Morgan inequalities.

4) The De Morgan negation is the split negation which is an evolution, thus we have that $\neg(\neg x \wedge \neg y) = \neg \neg x \vee \neg \neg y = x \vee y$ and $\neg(\neg x \vee \neg y) = \neg \neg x \wedge \neg \neg y = x \wedge y$. We have that $\neg x \geq \neg y$ implies $\neg \neg x \leq \neg \neg y$, so from $\neg \neg x = x$ we obtain also that $\neg x \geq \neg y$ implies $x \leq y$, thus the contraposition $x \leq y$ iff $\neg x \geq \neg y$. ■

Notice the naturality of this hierarchy, where from the general negation (the most weak negation) we reach from the antitonicity $x \leq y$ implies $\neg x \geq \neg y$ the stronger requirement of the contraposition ($x \leq y$ iff $\neg x \geq \neg y$) for De Morgan (the most strong) negation. Notice also how we pass from inequalities to stronger equality requirements. This is valid for any complete lattice where these negation exist (general negations always exist). If we impose the stronger requirements on complete lattices, as for example distributivity, then De Morgan negations coincide with the strongest classic Boolean negation (where for all $x \in X$, $\neg x \wedge x = 0$ and $\neg x \vee x = 1$).

The set-based semantics for the split negations (with Galois connections) can be given by the Birkhoff polarity operator $\lambda: (\mathcal{P}(X), \subseteq) \rightarrow (\mathcal{P}(X), \subseteq)^{OP}$, such that for any $U \in \mathcal{P}(X)$, $\lambda U = \{x \in X \mid \forall u \in U. (u, x) \in R\} = \{x \in X \mid \forall u. (u \in U \text{ implies } (u, x) \in R)\}$, which in a modal logic can be represented by " $\mathcal{M} \models_x \neg \phi$ iff $\forall u. (\mathcal{M} \models_u \phi \text{ implies } (u, x) \in R)$ " (the relation R is an incompatibility relation in Lemma 6). The set-based semantics for the constructive negation is obtained in the case where the incompatibility relation R is symmetric.

EXAMPLE 13. — The example for any complete lattice $(X, \leq)$ is a *general* negation given by $\neg x = \bigvee S_x$ where $S = \{z \in X \mid z \wedge x = 0\}$. It is well defined for complete lattices also when S_x is an *infinite* set, as for example in fuzzy logic where $X = [0, 1]$ is the closed interval of reals between 0 and 1, where for $\neg 0$ we have that $S_0 = \{z \in X \mid z \wedge 0 = 0\} = [0, 1]$. Notice that differently from distributive lattices, generally we have that in non distributive lattices $\neg x \notin S_x$, and $\neg x \wedge x \neq 0$ (we have that $\neg x \wedge x \in \{0, x\}$, i.e., $\neg x \wedge x = x$ if $x \leq \neg x$ (it cannot be $x > \neg x$), and $\neg x \wedge x = 0$ where x and $\neg x$ are not comparable; thus $\neg x \wedge x \leq x$). This negation in *distributive* lattices corresponds to the pseudocomplement negation where

for any $x \in X$, $x \wedge \neg x = 0$ (thus it cannot be used as paraconsistent negation) and is "constructive" $x \leq \neg\neg x$ but $\neg\neg x \not\leq x$. This negation in distributive algebras, where $x \vee \neg x = 1$ for any $x \in X$ is also valid, corresponds to the classic Boolean negation.

We are able to define also "non-constructive" general negation in any complete lattice by $\neg = \bigvee - \downarrow: X \rightarrow X$ where $-$ is the set complement in X , such that $\neg x = 0$ if $x = 1$; 1 otherwise. It is easy to verify that for any $x \in X$, $\neg x \leq \neg x$, $\neg x \vee x = 1$ and $\neg\neg x \leq x$. One example for distributive complete lattice with *paraconsistent* De Morgan negation is the fuzzy negation $\neg x = 1 - x$ for $x \in X = [0, 1]$ (where $x \wedge \neg x = \min(x, 1 - x) \neq 0$). In fact, it is additive and multiplicative ($\neg(x \wedge y) = 1 - \min(x, y) = \max(1 - x, 1 - y) = \neg x \vee \neg y$ and $\neg(x \vee y) = 1 - \max(x, y) = \min(1 - x, 1 - y) = \neg x \wedge \neg y$), thus it is self adjoint with $\neg\neg x = x$. Another paraconsistent De Morgan negation is the *bilattice negation* (Ginsberg, 1988): in Belnap's 4-valued bilattice it corresponds to the epistemic negation, such that $\neg f = t$, $\neg t = f$, $\neg \perp = \perp$ and $\neg \top = \top$, with $x \wedge \neg x \neq 0$ (while the pseudocomplement negation $\neg_t$ given in Example 8 (which is not a bilattice negation) is not paraconsistent). $\square$

The most simple definition for implication for bounded lattices which satisfies the Modus Ponens ($y \Rightarrow x, y \vdash x$) and Deduction Theorem ($z, y \vdash x$ iff $z \vdash y \Rightarrow x$) inference rules where $\vdash$ relation is equal to the $\leq$ partial order in the lattice (as in Definition 2, and where the logic entailment "preserves the truth", i.e., $x \vdash y$ iff $x \leq y$), is given by, $y \Rightarrow x = 1$ if $y \leq x$; x otherwise. But as we can see in this definition the value of y does not appear in the resulting value of the implication.

Both values x and y appear in the resulting value of the implication in the case when the lattice is modular (Dedekind's lattices) and satisfies $\neg x \wedge x = 0$ for every $x \in X$, when the Deduction Theorem from left to right does not hold.

The lattice is called *modular* when " $x \leq z$ implies $x \vee (y \wedge z) = (x \vee y) \wedge z$ ".

PROPOSITION 14. — *Let X be a complete modular lattice where $\neg x = \bigvee \{z \mid z \wedge x = 0\}$ and $\neg x \wedge x = 0$ for every $x \in X$, then we define the implication $\Rightarrow: X \times X \rightarrow X$ for any two elements $x, y \in X$ as follows:*

$$y \Rightarrow x =_{def} \begin{cases} 1 & \text{if } y \leq x \\ \neg y \vee x & \text{if } y > x \\ \neg y \vee (x \wedge y) & \text{otherwise.} \end{cases}$$

Then, $y \Rightarrow x = 1$ iff $y \leq x$, and hold the Modus Ponens "rule", $y \wedge (y \Rightarrow x) \leq x$ and half of the Galois connection, that is, " $z \leq y \Rightarrow x$ implies $z \wedge y \leq x$ ".

If we assume that the consequence pair relation $\vdash$ of this logic coincides with the partial order $\leq$ of this complete modular lattice (in Definition 2), then in this logic hold the Modus Ponens rule $y \wedge (y \Rightarrow x) \vdash x$ and the partially Deduction Theorem, that is, " $z \vdash y \Rightarrow x$ implies $z, y \vdash x$ ".

PROOF. — The definition of the implication is correct, that is, it is monotonic w.r.t the first argument and antitonic w.r.t the second argument. Moreover it is similar to the definition of the classic material implication.

Let us show that holds the Modus Ponens “rule”, that is, $(y \Rightarrow x) \wedge y \leq x$.

- 1.1 case when $y \leq x$: then, $(y \Rightarrow x) \wedge y = 1 \wedge y = y \leq x$.
- 1.2 case when $y > x$: then, $(y \Rightarrow x) \wedge y = (\neg y \vee x) \wedge y = (\text{by modular property}) = (\neg y \wedge y) \vee x = 0 \vee x = x$.
- 1.3 case when x and y are not comparable: then, $(y \Rightarrow x) \wedge y = (\neg y \vee (x \wedge y)) \wedge y = (\text{by the modular property and } x \wedge y \leq y) = (\neg y \wedge y) \vee (x \wedge y) = 0 \vee (x \wedge y) = x \wedge y \leq x$.

For the Deduction Theorem we have that if $z \leq y \Rightarrow x$, then:

- 2.1 case when $y \leq x$: then from, $z \leq y \Rightarrow x = 1$, we have $z = 1$, so that $z \wedge y = 1 \wedge y = y \leq x$.
- 2.2 case when $y > x$: then from, $z \leq y \Rightarrow x = \neg y \vee x$, we have (by the monotonicity of $\wedge$), $z \wedge y \leq y \wedge (\neg y \vee x) = (\text{by the modular property}) = (\neg y \wedge y) \vee x = 0 \vee x = x$.
- 2.3 case when x and y are not comparable: then from, $z \leq y \Rightarrow x = \neg y \vee (x \wedge y)$, we have (by the monotonicity of $\wedge$) that, $z \wedge y \leq y \wedge (\neg y \vee (x \wedge y)) = (\text{by the modular property}) = (\neg y \wedge y) \vee (x \wedge y) = 0 \vee (x \wedge y) = x \wedge y \leq x$.

Now for $y \Rightarrow x = 1$ and $z = 1$, and ($z \leq y \Rightarrow x$ implies $z \wedge y \leq x$), we obtain that ($y \Rightarrow x = 1$ implies $z \wedge y = 1 \wedge y = y \leq x$), and from the definition of $\Rightarrow$ we have that $y \leq x$ implies $y \Rightarrow x = 1$. Consequently, in this modular lattice we have $y \Rightarrow x = 1$ (or, alternatively, $\vdash y \Rightarrow x$) iff $y \leq x$.

Thus, if we assume that $\vdash$ is equal to $\leq$, then the MP “rule” $y \wedge (y \Rightarrow x) \leq x$ coincides with the Modus Ponens rule of inference and we obtain $\vdash (y \wedge (y \Rightarrow x)) \Rightarrow x$, i.e., that $(y \wedge (y \Rightarrow x)) \Rightarrow x$ is an axiom. Analogously, the Galois implication becomes half of the Deduction Theorem. ■

This class of modular lattices is strictly more general than the class of orthomodular lattices (Kalmbach, 1983), where it is also required that $\neg\neg x = x$ and $x \vee \neg x = 1$ for every x .

In any distributive lattice, where $\neg x \wedge x = 0$ is always satisfied for every $x \in X$, which is also modular, the definition of the implication given in Proposition 14 is always possible. But as we will see in what follows, in any distributive lattice the Modus Ponens and Deduction Theorem inference rules can be completely satisfied, and the logic implication can be defined precisely by this requirement.

4. Heyting’s and multi-modal extensions of distributive lattices

The meet and join operators correspond to the logic conjunction and disjunction. In order to have a full logic language we need also logic implication and logic negation.

For the class of distributive lattices we can introduce the negation operator as the pseudo-complement $\neg x = \bigvee \{z \mid z \wedge x = 0\}$, with $\neg x = 1$ iff $x = 0$, used in intuitionistic logics. In this section we will extend this distributive lattice with other unary modal operators and, based on them, two dual logic operators: the *implication* $\rightarrow: X \times X \rightarrow X$, and its dual *coimplication* operator $\rightsquigarrow: X \times X \rightarrow X$.

As we will see, the natural choice for the implication in complete lattices is the intuitionistic implication, based on the relative pseudo-complement, which always exists in complete distributive lattices. It is well known that the relative pseudo-complement $x \rightarrow y = \bigvee \{z \mid z \wedge x \leq y\}$ and classic implication satisfy the following left exponential commutative diagram,

where the arrow $(x \rightarrow y) \wedge x \leq y$ is the Modus Ponens inference rule $(x \rightarrow y), x \vdash y$, while for any $x \in X$ the additive modal operator $l_x = _ \wedge x : X \rightarrow X$ and its right adjoint multiplicative operator $r_x = x \rightarrow _ : X \rightarrow X$ define the Galois connection $l_x \dashv r_x$, that is, " $l_x(z) = z \wedge x \leq y$ iff $z \leq x \rightarrow y = r_x(y)$ " corresponding to the Deduction Theorem " $z, x \vdash y$ iff $z \vdash x \rightarrow y$ " (where the consequence relation $\vdash$ is equal to $\leq$, based on the "truth preservation" principle for valid logic derivations, in Definition 2).

The family of operators $\{l_x = _ \wedge x \mid x \in X\}$ may be considered as the family of existential modal operators, derived from the basic lattice meet operator $\wedge$, while r_x acts as its dual universal modal operator, but with $r_x \neq \neg l_x \neg$ and $l_x \neq \neg r_x \neg$.

The fundamental property (from the Galois connection when $z = 1$) of relative pseudo-complement is that " $x \leq y$ iff $x \rightarrow y = 1$ " (i.e., " $x \vdash y$ iff $\vdash x \rightarrow y$ ").

The right coexponential commutative diagram is dual (the arrows, that is, the partial ordering, are inverted), where the conjunction is replaced by disjunction (coconjunction) and the implication by its dual coimplication. The dual modal operators, for any $x \in X$ are the multiplicative modal operator $r_x^c = _ \vee x : X \rightarrow X$ and its left adjoint additive operator $l_x^c = x \rightsquigarrow _ : X \rightarrow X$ define the Galois connection $l_x^c \dashv r_x^c$, that is, " $l_x^c(y) = x \rightsquigarrow y \leq z$ iff $y \leq z \vee x = r_x^c(z)$ " corresponding to the coDeduction Theorem " $y \vdash z \vee x$ iff $x \rightsquigarrow y \vdash z$ ". The arrow $(x \rightsquigarrow y) \vee x \geq y$ corresponds to the coModus Ponens inference rule $y \vdash (x \rightsquigarrow y) \vee x$.

Thus, the coimplication is defined by $x \rightsquigarrow y = l_x^c(y) = \bigwedge \{z \mid y \leq z \vee x = r_x^c(z)\}$, such that “ $x \leq y$ iff $\neg(y \rightsquigarrow x) = 1$ ” (i.e., “ $x \vdash y$ iff $\vdash \neg(y \rightsquigarrow x)$ ”).

Thus $x \rightarrow y$ is an axiom iff $\neg(y \rightsquigarrow x)$ is an axiom, that is, $\vdash x \rightarrow y$ iff $\vdash \neg(y \rightsquigarrow x)$.

Notice that in Boolean algebras (distributive lattices with $\neg x \vee x = 1$ for any $x \in X$) we have that $x \rightarrow y = \neg x \vee y$ and $x \rightsquigarrow y = \neg x \wedge y$.

PROPOSITION 15. — *Each complete distributive lattice $(X, \leq)$ with bottom and top element $0, 1$ respectively, can be extended into Heyting algebra $B = (X, \leq, \wedge, \vee, \neg, \rightarrow, 0, 1)$ with implication $\rightarrow$ defined by $x \rightarrow y = \bigvee \{z \in X \mid z \wedge x \leq y\}$ and negation $\neg : X \rightarrow X$ defined by $\neg x = x \rightarrow 0$.*

The corresponding complete distributive lattice $(\mathcal{F}(X), \subseteq, \bigcap, \biguplus, \hat{\neg}, \rightsquigarrow, \{0\}, X)$ is the CANONICAL Heyting algebra with implication defined, for any $U, V \in \mathcal{F}(X)$, by $U \rightsquigarrow V = \biguplus \{Z \in \mathcal{F}(X) \mid Z \bigcap U \subseteq V\}$ and the negation operator (pseudo-complement) $\hat{\neg}, \hat{\neg}U = U \rightsquigarrow \{0\}$.

Then for any $x, y \in X$, $\hat{\neg} \downarrow x = \downarrow (\neg x)$ and $\downarrow x \rightsquigarrow \downarrow y = \downarrow (x \rightarrow y)$, that is, the following representation for any complete Heyting algebra is valid:

- 1) $\downarrow : (X, \leq, \wedge, \vee, \neg, \rightarrow, 0, 1) \simeq (\mathcal{F}(X), \subseteq, \bigcap, \biguplus, \hat{\neg}, \rightsquigarrow, \{0\}, X)$, and,
- 2) $\downarrow^+ : (X, \leq, \wedge, \vee, \neg, \rightarrow, 0, 1) \simeq (X^+, \subseteq, \bigcap, \bigcup, \hat{\neg}, \rightsquigarrow, \{0\}, \hat{X})$,

where $\hat{\neg}, \rightsquigarrow$ are obtained in the same way as above by substituting $\mathcal{F}(X)$ by X^+ , $\biguplus$ by $\bigcup$, and $\downarrow$ by $\downarrow^+$.

PROOF. — We obtain $\downarrow x \rightsquigarrow \downarrow y = \biguplus \{\downarrow z \in \mathcal{F}(X) \mid \downarrow z \bigcap \downarrow x \subseteq \downarrow y\} = \biguplus \{\downarrow z \in \mathcal{F}(X) \mid \downarrow (z \wedge x) \subseteq \downarrow y\} = \biguplus \{\downarrow z \in \mathcal{F}(X) \mid z \wedge x \leq y\} = \downarrow \bigvee \{z \in X \mid z \wedge x \leq y\} = \downarrow (x \rightarrow y) \in \mathcal{F}(X)$, that is, it is a stable set. Analogously, $\hat{\neg} \downarrow x = \downarrow (\neg x) \in \mathcal{F}(X)$ is a stable set. ■

Now we are able to introduce also the unary modal operators for these Heyting algebras, based on complete distributive lattices.

PROPOSITION 16. — *Each Heyting algebra $A = (X, \leq, \wedge, \vee, \neg, \rightarrow, 0, 1)$ has an invariant modal extension denominated banal Galois algebra $G = (X, \leq, \wedge, \vee, id, \neg, \rightarrow, 0, 1)$, isomorphic to $(\mathcal{F}(X), \subseteq, \bigcap, \biguplus, id_{\mathcal{F}(X)}, \hat{\neg}, \rightsquigarrow, \{0\}, X)$, where $\biguplus = \Gamma \bigcup$ and the identity function $id : X \rightarrow X$ is a self adjoint $id \dashv id$ modal operator correspondent to the universal modal operator $\Gamma : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$, whose reduction to the set of stable elements in $\mathcal{F}(X) \subset \mathcal{P}(X)$ is the identity operator $id_{\mathcal{F}(X)} = \downarrow \bigvee : \mathcal{F}(X) \rightarrow \mathcal{F}(X)$.*

We can extend a Heyting algebra A by any other unary additive modal operator $f : X \rightarrow X$, or $f : X \rightarrow X^{OP}$ (for modal negations in Lemma 12). So that we obtain the extended representation isomorphism for obtained Galois algebras:

- 1) $\downarrow : (X, \leq, \wedge, \vee, id, \neg, \rightarrow, f, 0, 1) \simeq (\mathcal{F}(X), \subseteq, \bigcap, \biguplus, id_{\mathcal{F}(X)}, \hat{\neg}, \rightsquigarrow, \hat{f}, \{0\}, X)$ where $\hat{f} = \downarrow f \downarrow^{-1} : \mathcal{F}(X) \rightarrow \mathcal{F}(X)$, and $\downarrow^{-1} = \bigvee$ is inverse homomorphism of $\downarrow$.

2) $\downarrow^+$: $(X, \leq, \wedge, \vee, id, \neg, \rightarrow, f, 0, 1) \simeq (X^+, \subseteq, \bigcap, \bigcup, id_{X^+}, \hat{\neg}, \hat{\rightarrow}, \hat{f}, \{0\}, \hat{X})$, obtained in the same way as in point 1, by substituting $\mathcal{F}(X)$ by X^+ , $\hat{\downarrow}$ by $\bigcup$, and $\downarrow$ by $\downarrow^+$.

PROOF. — Let $A = (X, \leq, \wedge, \vee, \neg, \rightarrow, 0, 1)$ be a Heyting algebra. Then the identity function $id : X \rightarrow X$ is monotone, i.e., $x \leq x'$ implies $id(x) = x \leq y = id(y)$, and preserves the meets: for any $x, y \in X$, $id(x \wedge y) = x \wedge y = id(x) \wedge id(y)$. Thus it is equivalent to the banal Galois algebra $G = (X, \leq, \wedge, \vee, id, \neg, \rightarrow, 0, 1)$.

The left adjoint to id is the operator $f : X \rightarrow X$, such that $f(x) = \bigwedge \{z \in Y \mid id(z) \geq x\} = x$, i.e., $f = id$, so that id is self adjoint.

By introducing this identity modal operator we obtain the Galois banal algebra *identical* to the original Heyting algebra. So this is an *invariant* modal extension for Heyting algebras, and can be considered as minimal modal extension for Heyting algebras. This fact explains that modal logic is intrinsic in any complete lattice $(X, \leq)$, and is based on the partial ordering $\leq$ of the poset X .

Let $f : X \rightarrow X$ be an additive modal operator (monotone which preserves all meets), then it has left adjoint $g : X \rightarrow X$, such that $g(x) \leq y$ iff $x \leq f(y)$. Then $x \leq y$ implies $f(x) \leq f(y)$, and $\downarrow f(x) \leq \downarrow f(y)$, i.e., $\downarrow f \downarrow^{-1} (\downarrow x) \leq \downarrow f \downarrow^{-1} (\downarrow y)$ with $\downarrow x \subseteq \downarrow y \in \mathcal{F}(X)$, that is, the monotone operator $\downarrow f \downarrow^{-1} : \mathcal{F}(X) \rightarrow \mathcal{F}(X)$ exists.

Thus, for any $U, V \in \mathcal{F}(X)$, $\downarrow f \downarrow^{-1} (U \cap V) = \downarrow f \downarrow^{-1} (\downarrow (x \wedge y)) = \downarrow f(x \wedge y) = \downarrow f(x) \cap \downarrow f(y) = \downarrow f \downarrow^{-1} (\downarrow x) \cap \downarrow f \downarrow^{-1} (\downarrow y)$. Consequently $\downarrow f \downarrow^{-1}$ preserves the meets, and both with a monotone property it means that it is an additive unary modal operator in a canonical Heyting algebra.

Moreover, from $g(x) \leq y$ iff $x \leq f(y)$ we obtain $\downarrow g \downarrow^{-1} (\downarrow x) \leq \downarrow y$ iff $\downarrow x \leq \downarrow f \downarrow^{-1} (\downarrow y)$, that is $\downarrow g \downarrow^{-1}$ is the left adjoint of the $\downarrow f \downarrow^{-1}$, with $\downarrow g \downarrow^{-1} = \hat{\neg} \downarrow f \downarrow^{-1} \hat{\neg}$. Consequently $\downarrow f \downarrow^{-1}$ is the universal while $\downarrow g \downarrow^{-1}$ is the existential modal operator in the canonical Heyting (i.e., Galois) algebra $\mathcal{F}(X)$. ■

EXAMPLE 17. — In Belnap's bilattice the conflation $-$ is a monotone function which preserves all finite meets (and joins) w.r.t the lattice $(\mathcal{B}, \leq_t)$, thus it is an universal (and existential, i.e., $f = g = -$, because $- = \neg_t - \neg_t$) modal many-valued operator: "it is believed that", which extends the 2-valued belief of the autoepistemic logic as follows:

- 1) if A is true then "it is believed that A ", i.e., $-A$, is true;
- 2) if A is false then "it is believed that A " is false;
- 3) if A is unknown then "it is believed that A " is inconsistent: it is really inconsistent to believe in something that is unknown;
- 4) if A is inconsistent (that is *both* true and false) then "it is believed that A " is unknown: really, we can not tell anything about believing in something that is inconsistent.

This belief modal operator can be used to define the *epistemic*(bilattice) negation $\neg$, as a composition of strong negation $\neg_t$ and this belief operator, *i.e.*, $\neg = \neg_t \circ$. That is, the epistemic negation is negation as a modal operator (Došen, 1986) and as a paraconsistent negation (Béziau, 2002). We can introduce also Moore's *autoepistemic* operator (Ginsberg, 1988), $\mu : \mathcal{B} \rightarrow \mathcal{B}$, for a Belnap's bilattice, defined by $\mu(x) = t$ if $x \in \{\top, t\}$; f otherwise.

It is easy to verify that it is monotone w.r.t the $\leq_t$, that is multiplicative ($\mu(x \wedge y) = \mu(x) \wedge \mu(y)$) and $\mu(t) = t$ and additive ($\mu(x \vee y) = \mu(x) \vee \mu(y)$ and $\mu(f) = f$), consequently also it is a selfadjoint (contemporary universal and existential) modal operator, $\mu = \neg_t \mu \neg_t$. Notice that differently from the belief modal operator $\neg$, Moore's modal operator μ is not surjective.

Such an autoepistemic intuitionistic logic, based on the 4-valued Belnap's bilattice, $(\mathcal{B}, \leq_t, \wedge, \vee, \neg_t, \rightarrow, \neg, \mu, f, t)$ can be usefully used for logic programming with incomplete and partially inconsistent information (Majkić, 2005a) (in such logic we use only the epistemic negation $\neg = \neg_t \circ$). $\square$

We have seen that the for distributive lattices we can define the "strict" implication (relative pseudo-complement), such that for any two logic formulae ϕ and ψ holds that $\phi \rightarrow \psi = \Box(\phi \Rightarrow_c \psi)$ where $\Rightarrow_c$ is the classical extensional implication such that $\phi \Rightarrow_c \psi = \neg_c \phi \vee_c \psi$ (here $\neg_c$ and $\vee_c$ denote the classic logic negation and disjunction respectively) and $\Box$ is not necessarily truth-functional in this distributive lattice. This "strict" implication is the *intuitionistic* logic implication, where all axioms of the classic propositional logic are satisfied, except the axiom for disjunction $\neg x \vee x = (x \rightarrow 0) \vee x$.

Now we will investigate the subclass of distributive complete lattices, where there exists the *algebraic truth-functional* counter-part $r_{\Box} : X \rightarrow X$, of the logic modal necessary operator $\Box$, and is an identity, so that $\neg x \vee y$ (*i.e.*, $(x \rightarrow 0) \vee y$) can be used as implication $x \Rightarrow y$. In what follows, we will show that this "or-implication" satisfies the Modus Ponens, but generally does not satisfy the Deduction Theorem,

PROPOSITION 18. — *The or-implication $x \Rightarrow y =_{def} \neg x \vee y$ satisfies the Modus Ponens but not the Deduction Theorem w.r.t the entailment in Definition 2, that is, for any $x \in X$, $r = \neg x \vee _ : X \rightarrow X$ is a monotone multiplicative operator, thus right adjoint of the operator $l : X \rightarrow X$, with $l \neq _ \wedge x$.*

PROOF. — For the or-implication we have that, from the distributivity, $(x \Rightarrow y) \wedge x = (\neg x \vee y) \wedge x = (\neg x \wedge x) \vee (y \wedge x) = 0 \vee (y \wedge x) = y \wedge x \leq y$, that is the Modus Ponens rule holds.

The operator $r = \neg x \vee _$ is monotone because $\vee$ is monotone for both arguments. And, from the distributivity, $r(y \wedge z) = \neg x \vee (y \wedge z) = (\neg x \vee y) \wedge (\neg x \vee z) = r(y) \wedge r(z)$, and $r(1) = \neg x \vee 1 = 1$, thus r is multiplicative, so that its left adjoint operator l is defined for any $y \in X$, by $l(y) = \bigwedge \{z \mid y \leq r(z) = \neg x \vee z\}$, with $l(y) \neq y \wedge x$. ■

The Deduction Theorem can not be satisfied because there exist $x, y \in X$ such that $\neg x \vee y < x \rightarrow y = \bigvee \{z \mid z \wedge x \leq y\}$, and if we substitute the relative

pseudo-complement $x \rightarrow y$ by "or-implication" $\neg x \vee y$ we are not able to obtain the commutative exponential diagram as in the case of a relative pseudo-complement.

Thus, in order to satisfy the Deduction Theorem, the or-implication $r(y) = \neg x \vee y$ must be a right adjoint to $l = _ \wedge x$, that is, must be equal to the relative pseudo-complement $x \rightarrow y$. In the next proposition we will define the subclass of complete distributive lattices where such a condition is satisfied.

PROPOSITION 19. — *Each complete distributive lattice $(X, \leq)$ with the set of join-irreducible elements $\widehat{X} - \{0\}$ equal to the set of atoms $\underline{X}$, is a complete distributive lattice, where for every $x, y \in X$, $x \rightarrow y = \neg x \vee y$ and viceversa.*

PROOF. — First of all we will show that $\underline{X} = \widehat{X} - \{0\}$ is a sufficient condition for $x \rightarrow y = \neg x \vee y$. Each atom is join-irreducible, thus $\underline{X} \subseteq \widehat{X} - \{0\}$. Suppose that $\underline{X} \subset \widehat{X} - \{0\}$. Then, from the Birkhoff theorem for distributive lattices, $\bigvee \underline{X} < \bigvee(\widehat{X} - \{0\}) = 1$, and for any atom $a \in \underline{X}$, $\neg a < \bigvee \underline{X}$ (because $a < \bigvee \underline{X}$), so that $\neg a \vee a \leq \bigvee \underline{X} \vee a = \bigvee \underline{X} < 1$, while $a \rightarrow a = 1$, i.e., we obtain that $\neg a \vee a \neq a \rightarrow a$, which is a contradiction. Thus $\underline{X} = \widehat{X} - \{0\}$ must be correct.

Let us assume that $\underline{X} = \widehat{X} - \{0\}$, and show that for every $x, y \in X$, $x \rightarrow y = \neg x \vee y$ must hold. We consider the following cases:

1. case when $x \leq y$: in that case $x \rightarrow y = 1$, and, from the Birkhoff isomorphism (Proposition 9), we have that for unique subsets $S_x, S_y \subseteq \underline{X}$ such that $x = \bigvee S_x$, $y = \bigvee S_y$, must hold $S_x \subseteq S_y$, i.e., $S_y = S_x \cup \Delta$, and $\neg S_x \cup S_y = \neg S_x \cup (S_x \cup \Delta) = \underline{X} \cup \Delta = \underline{X}$. Thus $x \rightarrow y = 1 = \bigvee \underline{X} = \bigvee(\neg S_x \cup S_y) = \bigvee(\neg S_x) \vee (\bigvee S_y) = \neg x \vee y$.
2. case when $x \not\leq y$ and $\neg x = 0$: Let us show that in this case $x = 1$ must be correct. Suppose that $\exists x \neq 1. \neg x = 0$, then $\nexists y \boxtimes x. (y \neq 0 \text{ and } y = \neg x)$, that is $x = \bigvee \underline{X}$ but $1 = \bigvee \underline{X}$, and from the Birkhoff bijection (isomorphism) must be $x = 1$, so that $x \rightarrow y = 1 \rightarrow y = \bigvee \{z \mid z \wedge 1 = z \leq y\} = y = \neg x \vee y$. This case holds when $x > y$. In fact, $\neg x = 0$ must be correct, otherwise the sublattice $0 < \neg x < \neg x \vee x$, $0 < y < x < \neg x \vee x$ is a N5 diagram which can not be a sublattice of modular (thus distributive) lattice $(X, \leq)$.
3. case when $x \boxtimes y$ and $\neg x \neq 0$: it is enough to consider only $x, y \notin \{0, 1\}$ which are particular subcases of the cases above. It is easy to verify that for any distributive lattice it holds that
 (**) $x \rightarrow y \geq \neg x \vee y$ (because $x \rightarrow y \geq \neg x$ and $x \rightarrow y \geq y$, from $y \in \{z \mid z \wedge x \leq y\}$).
 Suppose that $x \rightarrow y > \neg x \vee y$, then there are the following subcases:
 - 3.1 when $y > \neg x = \bigvee \{z \mid z \wedge x = 0\}$: then $\neg x \vee y = y$, so that $y \geq x$ (otherwise, $\neg x = y$) which is in contradiction with $x \boxtimes y$.
 - 3.2 when $y < \neg x$: then we obtain the following N5 sublattice, $0 < x < x \vee \neg x$, $0 < y < \neg x < x \vee \neg x$ which can not hold for modular (thus distributive) lattices.

3.3 when $y = \neg x$: then $x \rightarrow y < 1$ (because $x \not\leq y$). Cannot be $x \leq x \rightarrow y$. If so, then $x \wedge (x \rightarrow y) = x$ (Modus Ponens in distributive lattices), i.e., $y = \neg x \geq x$ which is in contradiction with $x \bowtie y$. Cannot be $x > x \rightarrow y$ (if so, then $\neg x \vee y = y \geq x \wedge (x \rightarrow y) = x \rightarrow y$ which is in contradiction with hypothesis $x \rightarrow y > \neg x \vee y$). Thus we obtain the N5 sublattice $0 < x < x \vee (x \rightarrow y)$, $0 < \neg x \vee y < x \rightarrow y < x \vee (x \rightarrow y)$ which cannot hold for modular (thus distributive) lattices.

3.4 when $y \bowtie \neg x$: then, cannot be $x \leq x \rightarrow y$ (if so, then $x = x \wedge (x \rightarrow y) \leq y$ (from definition of $x \rightarrow y$), which is in contradiction with $x \bowtie y$. Suppose that $x \geq x \rightarrow y$, then $x \rightarrow y = x \wedge (x \rightarrow y) \leq y$ (from definition of $x \rightarrow y$), and from $x \rightarrow y \geq y$ we obtain that $x \rightarrow y = y$. Thus, $\neg x \vee y \geq y = x \rightarrow y$, which is in contrast with (**) $x \rightarrow y \geq \neg x \vee y$. Consequently, $x \bowtie (x \rightarrow y)$, and $x \vee (x \rightarrow y) > x \rightarrow y$ and $x \vee (x \rightarrow y) > x$ (notice that from $x \bowtie y$ we have that $x \wedge y < x$, and $\neg x \vee y > x \wedge (\neg x \vee y) = (x \wedge \neg x) \vee (x \wedge y) = 0 \vee (x \wedge y) = x \wedge y$).

Thus we obtain the N5 sublattice $x \wedge y < x < x \vee (x \rightarrow y)$, $x \wedge y < \neg x \vee y < x \rightarrow y < x \vee (x \rightarrow y)$ which cannot hold for modular (thus distributive) lattices. ■

REMARK. — If for every $x, y \in X$, $x \rightarrow y = \neg x \vee y$ then $\neg x \vee x = x \rightarrow x = 1$, and from the general property for distributive lattices, $\neg x \wedge x = 0$, we obtain that the distributive lattices with the property defined in Proposition 19 are Boolean algebras. □

EXAMPLE 20. — The classic logic with 2-valued lattice ($\mathbf{2} = \{0, 1\}, \leq$) is a Boolean algebra: the set of atoms is equal to the set of join-irreducible elements $\{1\}$. The cartesian product $X = \mathbf{2}^2 = \mathbf{2} \times \mathbf{2} = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$, with $(x, y) \leq (v, w)$ iff $x \leq v$ and $y \leq w$, is a Boolean algebra with the set of atoms (i.e., join-irreducible elements) $\underline{X} = \{(0, 1), (1, 0)\}$ and negation defined by $\neg(x, y) = (\neg x, \neg y)$. It is isomorphic to the Belnap's 4-valued lattice $\mathcal{B}_4$ where $f \mapsto (0, 0)$, $t \mapsto (1, 1)$, $\perp \mapsto (0, 1)$ and $\top \mapsto (1, 0)$.

It is easy to verify that every lattice $X_n = (\mathbf{2}^K, \leq)$, where $K \geq 2$ and $(x_1, \dots, x_k) \leq (y_1, \dots, y_k)$ if for all $1 \leq i \leq K$, $x_i \leq y_i$, is a Boolean algebra where negation is defined by $\neg(x_1, \dots, x_k) = (\neg x_1, \dots, \neg x_k)$. But we have also that in each Boolean algebra $(\mathcal{P}(S), \subseteq, \cap, \cup, -)$ where $-$ is the set complement, the set S is exactly the set of atoms). Thus we have the following isomorphisms of lattices: $(\mathbf{2}, \leq) \simeq (\mathcal{P}(\{a_1\}), \subseteq)$, $(\mathcal{B}_4, \leq) \simeq (\mathbf{2}^2, \leq) \simeq (\mathcal{P}(\{a_1, a_2\}), \subseteq)$, ..., $(\mathbf{2}^K, \leq) \simeq (\mathcal{P}(\{a_1, \dots, a_K\}), \subseteq)$, $K \geq 2$. □

5. Autoreferential Kripke-style semantics for many-valued logics

We will use the invariant and intrinsic modal properties of complete lattices and Heyting algebras in order to obtain a basic (or canonical) Kripke structure for any such algebra in a general way.

By $\mathbf{2} = \{0, 1\} \subseteq X$ we denote the set of classic logic values (false and true respectively), and by H the Herbrand base, *i.e.*, the set of all ground atoms in a many-valued predicate logic $\mathcal{L}$, with $H \subseteq \mathcal{L}$.

Given a many-valued interpretation $I : H \rightarrow X$, we denote by $\bar{I} : \mathcal{L} \rightarrow X$ its unique extension to all formulae, obtained inductively as follows: for any ground atom $r(c_1, \dots, c_n) \in H$, $\bar{I}(r(c_1, \dots, c_n)) = I(r(c_1, \dots, c_n))$; for any two formulae $\phi, \psi \in \mathcal{L}$, $\bar{I}(\phi \wedge \psi) = \bar{I}(\phi) \wedge \bar{I}(\psi)$; $\bar{I}(\phi \vee \psi) = \bar{I}(\phi) \vee \bar{I}(\psi)$; $\bar{I}(\phi \Rightarrow \psi) = \bar{I}(\phi) \multimap \bar{I}(\psi)$; $\bar{I}(\neg \phi) = \neg \bar{I}(\phi)$, and for any algebraic additive modal operator o_i , $\bar{I}(\Diamond_i \phi) = o_i \bar{I}(\phi)$, where $\Diamond_i$ is a logic symbol for an existential modal operator (and $\Box_i$ is its correspondent universal logic modal operator).

Based on the general Representation Theorem for many-valued logic in (Majkić, 2006d), and the *autoreferential assumption*, we assume that the set of possible worlds of relational Kripke frames is the set of algebraic truth-values in a complete lattice X of this many-valued logic, or, alternatively, the set of join-irreducible elements $\hat{X}$ (Majkić, 2006d) for distributive complete lattices with normal additive modal operators only. In what follows we will present the first approach, which is valid also for *non-distributive* lattices (for logics where the implication cannot be a relative pseudo-complement).

From the formal mathematical point of view, we can consider the stable sets in $\mathcal{F}(X)$ as 'stable characteristic functions' in $\mathbf{2}^X$ ($\mathbf{2}^X$ denotes the set of all functions from X to $\mathbf{2}$), that is, the functions whose image is a stable set (element in $\mathcal{F}(X)$). Let $\mathcal{F}_X \simeq \mathcal{F}(X)$ denote such a set of stable functions, bijective to $\mathcal{F}(X)$, whose elements are functions (*i.e.*, higher-order algebraic truth-values (Majkić, 2006c)). Than any many-valued valuation v , of a propositional logic $\mathcal{L}$ with a set of propositional letters in P , based on this set of higher-order truth-values is a mapping $v : P \rightarrow \mathcal{F}_X \subset \mathbf{2}^X$. So that we are able to define the valuation $V : P \times X \rightarrow \mathbf{2}$, such that for any $p \in P, x \in X$, $V(p, x) = v(p)(x) \in \mathbf{2}$, which is identical to Kripke valuations used for standard 2-valued Kripke models of propositional modal logics, if we accept (by autoreferential assumption) for the set of *possible worlds* the set of algebraic truth-values X .

The philosophical autoreferential assumption is based on the consideration that each possible world represents a level of *credibility*, so that only the propositions with the right logic value (*i.e.*, level of credibility) can be accepted by this world.

Now we will define the accessibility relation for any given modal operator o_i .

DEFINITION 21. — Let $o_i : X \rightarrow X$ be a monotonic operator with $S = \{(o_i(x), x) \mid x \in X\}$, and the set $S_m \subseteq S$ obtained by elimination of each element $(x, y) \in S$ if there exists another element $(x, z) \in S$ such that $z \leq y$.

Then we define the accessibility relation for o_i by

$$\mathcal{R}_i = S_m \cup \{(x, y) \mid \nexists (x, v) \in S_m\} \text{ where } \begin{cases} y = 0 & \text{if } x = 0; \text{ otherwise} \\ y = z & \text{if } \exists (v, z) \in S_m. (x \leq v). \end{cases}$$

Notice that if $x \neq 0$ and $\nexists y \in X.(x \leq o_i(y))$ then $x \notin \pi_1(\mathcal{R}_i)$, where π_1 is the first projection.

EXAMPLE 22. — In the case of the believe (conflation) modal operator — in Belnap's bilattice, we obtain that $\mathcal{R}_- = S_m = S = \{(f, f), (\top, \perp), (\perp, \top), (t, t)\}$, while for the autoepistemic Moore's operator μ , $S = \{(f, f), (f, \perp), (t, \top), (t, t)\}$, $S_m = \{(f, f), (t, \top)\} \subset S$ and $\mathcal{R}_\mu = \{(f, f), (t, \top), (\top, \top), (\top, t), (\perp, \top), (\perp, t)\}$. $\square$

DEFINITION 23. — Let $\tilde{o}_i : (X, \leq) \rightarrow (X, \leq)^{OP}$ be a split negation operator. We define the binary relation S as follows:

$$(a) S = \{(x, \tilde{o}_i(x)) \mid x \in X\},$$

(b) For every $y \in X$ substitute S by the relation $(S - S_y) \cup \{(\bigvee \pi_1 S_y, y)\}$, where $S_y = \{(x, y) \mid (x, y) \in S\} \subset S$ and π_1 is the first projection.

Then we define the relation $\widetilde{\mathcal{R}}_i$ for $\tilde{o}_i$ by: $S \subseteq \widetilde{\mathcal{R}}_i$ and if $(x, y) \in \widetilde{\mathcal{R}}_i$ and $x' \leq x$, $y' \leq y$ then $(x', y') \in \widetilde{\mathcal{R}}_i$.

The split negation operators $\tilde{o}_i$ correspond to the monotone additive operators $\tilde{o}_i : X \rightarrow X^{OP}$, and the correspondent relations $\widetilde{\mathcal{R}}_i$ are "perp" (Perpendicular or incompatibility) relations, introduced by Goldblatt in (Goldblatt, 1974).

Notice that the point (b) for each $y \in X$ substitutes the equivalence class S_y by the single pair (x, y) where $x = \bigvee \pi_1 S_y = \bigvee \{z \mid (z, y) \in S_y\}$, so that the following lemma holds (by π_2 we denote the second projection):

LEMMA 24. — Let S be a binary relation in Definition 23 for a split negation operator $\tilde{o}_i = \neg : (X, \leq) \rightarrow (X, \leq)^{OP}$. Then $\neg_r : (S_1, \leq) \rightarrow (S_2, \leq)^{OP}$, where $S_1 = \pi_1 S \subseteq X$, $S_2 = \pi_2 S \subseteq X$ is the bijective domain reduction of the split operator $\neg$ (i.e., for each $x \in S_1$, $\neg_r x = \neg x$), with its inverse $\neg_r^{-1} : (S_2, \leq)^{OP} \rightarrow (S_1, \leq)$, such that

- 1) for any two $x, x' \in S_1$, $x < x'$ iff $\neg x > \neg x'$,
- 2) for any $x \in X$ we have that $x' = \neg_r^{-1} \neg x \in S_1$, such that $\neg x = \neg_r x'$ with $x \leq x'$.

PROOF. — Let us show that $\neg_r$ is a domain reduction of $\neg$. In fact, after the actions defined in point (b) of Definition 23, we have that the cardinalities of S_1 and S_2 are equal, and for any $y \in S_2$ we have the unique pair $(x, y) \in S$ where $x = \bigvee \pi_1 S_y = \bigvee \{z \mid (z, y) \in S_y\}$. Let us show that $\neg_r x = y$: in fact, by definition $\neg_r x =_{def} \neg x = \neg \bigvee \{z \mid (z, y) \in S_y\} = \bigwedge \{\neg z \mid (z, y) \in S_y\}$ (from the act that $\neg$ is a split negation, thus an additive operator) $= y$ (because from the definition of the equivalence class S_y in Definition 23 we have that for each $(z, y) \in S_y$, $\neg z = y$). Thus, the definition of $\neg_r$ as the domain reduction of $\neg$ is correct. We have also that for any two $(x, y), (x', y') \in S$ we have that $x \neq x'$ iff $y \neq y'$, thus $\neg_r$ is a bijection, with inverse mapping $\neg_r^{-1} : S_2 \rightarrow S_1$ defined by, for any $y \in S_2$, $\neg_r^{-1}(y) = x$, where $(x, y) \in S$. Thus, $\neg_r^{-1} \neg_r = id_{S_1}$ and $\neg_r \neg_r^{-1} = id_{S_2}$. Consequently, also $\neg_r^{-1}$ is monotone, i.e., for any

two $\neg_r x, \neg_r x' \in S_2$, if $\neg_r x >^{OP} \neg_r x'$ (i.e., $\neg_r x < \neg_r x'$, then $\neg_r^{-1}(\neg_r x) = x > \neg_r^{-1}(\neg_r x') = x'$, and from the monotonicity of $\neg$ (and its domain reduction $\neg_r$) we obtain that the point (1) of this lemma holds.

Point (2): from the definition of S , x' is the join of all elements in $\pi_1 S_y$ in an equivalence class S_y , where $y = \neg x$, with $x \in \pi_1 S_y$, so that $x \leq x'$. ■

EXAMPLE 25. — Finite cases: In the case of the bilattice (epistemic) negation operator $\neg$ in Belnap's bilattice, which is paraconsistent De Morgan negation (thus split negation also), we obtain:

$S = \{(f, t), (\top, \top), (\perp, \perp), (t, f)\}$, with $S_1 = S_2 = X = \mathcal{B}_4$, $\neg_r = \neg_r^{-1} = \neg$, and $\widetilde{\mathcal{R}}_{\neg} = \{(f, f), (f, \perp), (f, \top), (f, t)(\perp, \perp), (\perp, f), (\top, \top), (\top, f), (t, f)\}$.

Infinite cases: In the case of the fuzzy logic, $\neg x = 1 - x$, it is a paraconsistent De Morgan negation thus also split negation, with $S = \{(x, 1 - x) \mid x \in X = [0, 1]\}$. Thus, from Definition 23, we obtain a symmetric *infinite* incompatibility relation $\widetilde{\mathcal{R}}_{\neg}$ as in the example above: it must be so because both negations are selfadjoint, so that this incompatibility relation must be symmetric. Other cases for paraconsistent split negations are given in belief and confidence level logics and their fuzzy extensions (Majkić, 2005b). □

Now we are able to define the *many-valued relational* Kripke-style semantics (the 2-valued Kripke-style semantics is presented in (Majkić, 2006e)) for a predicate modal logic $\mathcal{L}$ based on the modal Heyting algebras in Proposition 16:

DEFINITION 26. — Given a predicate modal logic $\mathcal{L}$, based on the modal Heyting algebra in Proposition 16, we define the Kripke model for this logic $\mathcal{M} = (K, S, V)$ with the frame $K = \langle (X, \leq), \{\mathcal{R}_i, \widetilde{\mathcal{R}}_i\} \rangle$, where each $\mathcal{R}_i, \widetilde{\mathcal{R}}_i, i = 1, 2, \dots$, is the accessibility relation for existential (additive) modal operators o_i and negation modal operator $\widetilde{o}_i$ respectively, given in Definition 21, with a decreasing valuation $V : X \times P \rightarrow \bigcup_{n < \omega} 2^{S^n}$, such that for any predicate symbol $r \in P$ with arity n , and tuple of constants $(c_1, \dots, c_n) \in S^n$ (i.e., ground atom $r(c_1, \dots, c_n) \in H$), there exists a particular value $y \in X$, such that $\forall x \in X (V(x, r)(c_1, \dots, c_n) = 1 \text{ iff } x \leq y)$.

Then, for any world $x \in X$ and assignment g , we define the many-valued satisfaction relation for a formula $\phi \in \mathcal{L}$, denoted by $\mathcal{M} \models_{x,g} \phi$, as follows:

- 1) $\mathcal{M} \models_{x,g} r(x_1, \dots, x_n) \text{ iff } V(x, r)(g(x_1), \dots, g(x_n)) = 1$,
- 2) $\mathcal{M} \models_{x,g} \phi \wedge \psi \text{ iff } \mathcal{M} \models_{x,g} \phi \text{ and } \mathcal{M} \models_{x,g} \psi$,
- 3) $\mathcal{M} \models_{x,g} \phi \vee \psi \text{ iff } \forall y ((y \leq x \text{ and not } \mathcal{M} \models_{y,g} \phi) \text{ implies } \mathcal{M} \models_{y,g} \psi)$,
- 4) $\mathcal{M} \models_{x,g} \phi \Rightarrow \psi \text{ iff } \forall y ((y \leq x \text{ and } \mathcal{M} \models_{y,g} \phi) \text{ implies } \mathcal{M} \models_{y,g} \psi)$,
- 5) $\mathcal{M} \models_{x,g} \neg \phi \text{ iff } \mathcal{M} \models_{x,g} \phi \Rightarrow \emptyset$, where $\emptyset$ is a contradiction formula,
- 6) $\mathcal{M} \models_x \Diamond_i \phi \text{ iff } \exists y ((x, y) \in \mathcal{R}_i \text{ and } \mathcal{M} \models_y \phi)$, for any monotone modal operator $\Diamond_i$.
- 7) $\mathcal{M} \models_x \sim_i \phi \text{ iff } \forall y (\mathcal{M} \models_y \phi \text{ implies } (y, x) \in \widetilde{\mathcal{R}}_i)$, for any modal negation operator $\sim_i$, so that $\|\sim_i \phi\| = \lambda \|\phi\| = \{x \mid \forall y \in \|\phi\|. (y, x) \in \widetilde{\mathcal{R}}_i\}$.

By definition $\mathcal{M}$ is a Kripke-style model of a logic $\mathcal{L}$ if it satisfies all formulae in $\mathcal{L}$.

The reason that we use both negations, the pseudocomplement $\neg$ and modal (weak) negations $\sim_i$ is justified by necessity to consider also the *paraconsistent* many-valued logics in which $\neg$ can not be used as paraconsistent negation (in Proposition 30).

Notice that in the world $x = 0$ (the bottom element in X) every formula $\phi \in F(\mathcal{L})$ is satisfied: because of that we will denominate this world by *trivial world*. The semantics for implication is similar (but different) to the Kripke style definition for intuitionistic implication (where the time sequence for the accessibility relation is used, upward (in the future) closed and based on the principle "persistence of truth in time" (Kripke, 1965)). Otherwise, the definition for disjunction is not standard. In fact, it holds that $(\mathcal{M} \models_{x,g} \phi \text{ or } \mathcal{M} \models_{x,g} \psi)$ implies $\mathcal{M} \models_{x,g} \phi \vee \psi$, but *not vice versa*. It results from the fact that the disjunction in the canonical representation is the set operator $\uplus$, (generally) different from the standard set union $\cup$.

The satisfaction for disjunction can be given alternatively by

3.a $\mathcal{M} \models_{x,g} \phi \vee \psi$ iff $\exists y, z (x \leq y \vee z \text{ and } \mathcal{M} \models_{y,g} \phi \text{ and } \mathcal{M} \models_{z,g} \psi)$.

Consequently the conjunction disjunction and implication are defined independently as in intuitionistic logic (McKinsey *et al.*, 1948). In fact, the Definition 26 for a Kripke model of many-valued modal logics is similar but *not equal* (downward closed instead of upward closed hereditary subsets, and the worlds are not the time-points as in Kripke interpretation but the autoreferential *truth-levels of credibility*) to that of Kripke-style intuitionistic logic because not only the implication (as in (McKinsey *et al.*, 1948)) but also the *disjunction is intensional*: it is based on Lewis's approach, as discussed in Remark after Proposition 7, where he distinguish the standard or *extensional* logic connectives $\wedge_c, \vee_c, \Rightarrow_c$, from the *intensional* logic connectives $\wedge, \vee, \Rightarrow$, such that $x \odot y = \Box(x \odot_c y)$, for $\odot \in \{\wedge, \vee, \Rightarrow\}$ where $\Box$ is "necessarily" a universal logic modal operator corresponding to the algebraic modal operator $\Gamma : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$.

The accessibility relation for the multiplicative modal operator "necessary" $\Box$ is the partial order $\geq = \leq^{-1}$ in Definition 26. For its left adjoint (additive) dual modal operator $\Diamond, \Diamond \dashv \Box$, such that the Galois connection $\Diamond x \leq y$ iff $x \leq \Box y$ holds, the accessibility relation can be obtained by the "gaggle" theory of Dunn (Dunn, 1991; Dunn, 1993), for this simple Galois connection with the following traces and tonicities:

function	tonicity	trace
$\Box$	+	$+\mapsto +$
$\Diamond$	+	$-\mapsto -$

From (Dunn, 1991) it holds that the binary accessibility relations of these two unary adjoint operators are mutually inverse, *i.e.*, $\mathcal{R}_\Diamond = (\overline{\mathcal{R}_\Box})^{-1} = (X \times X - \mathcal{R}_\Box)^{-1} = X \times X - \leq$.

It is intuitive also because these two operators are usually given as two mappings $\Diamond : (X, \leq) \rightarrow (X, \leq)$ and $\Box : (X, \leq)^{OP} \rightarrow (X, \leq)^{OP}$, where $(X, \leq)^{OP}$ is a poset with inverted arrows (ordering).

We will consider that, the downward closure for the satisfaction holds for each ground formula ϕ/g , that is if $\mathcal{M} \models_{x,g} \phi$ than for every $y \leq x$ holds $\mathcal{M} \models_{y,g} \phi$: it is satisfied for all ground atoms, directly from the definition of the mapping V in Definition 26, and will be proven in Theorem 28 bellow.

Thus, we can use the standard Kripke definition for the satisfaction relation for these intensional connectives, that is:

1. for the principal modal operator $\Box$:

$\mathcal{M} \models_{x,g} \Box r(x_1, \dots, x_n)$ iff $\forall y (y \leq x \text{ implies } \mathcal{M} \models_{y,g} r(x_1, \dots, x_n))$ iff $\mathcal{M} \models_{x,g} r(x_1, \dots, x_n)$. Thus, it is invariant for ground atoms. Moreover, for any ground formula ϕ/g such that $\|\phi/g\| = \{x \mid \mathcal{M} \models_{x,g} \phi\} \in \mathcal{F}(X)$, i.e., $\|\phi/g\| = \downarrow \alpha$ for some $\alpha \in X$, we have that $\mathcal{M} \models_{x,g} \Box \phi$ iff $\forall y (y \leq x \text{ implies } \mathcal{M} \models_{y,g} \phi)$ iff $\forall y (y \leq x \text{ implies } y \in \downarrow \alpha)$ iff $x \leq \alpha$ iff $\mathcal{M} \models_{x,g} \phi$. That is, $\Box$ is an identity.

But if $\|\phi/g\| \notin \mathcal{F}(X)$ then $\Box$ is not an identity, but $\|\Box \phi/g\| = \Gamma \|\phi/g\| \in \mathcal{F}(X)$.

Also in the case of the dual, existential, modal operator $\Diamond$ we have for that any ground formula ϕ/g such that $\|\phi/g\| = \{x \mid \mathcal{M} \models_{x,g} \phi\} \in \mathcal{F}(X)$, i.e., $\|\phi/g\| = \downarrow \alpha$ for some $\alpha \in X$, holds $\mathcal{M} \models_{x,g} \Diamond \phi$ iff $\exists y (x \leq y \text{ and } \mathcal{M} \models_{y,g} \phi)$ iff $\exists y (x \leq y \text{ and } y \in \downarrow \alpha)$ iff $x \leq \alpha$ iff $\mathcal{M} \models_{x,g} \phi$. That is, $\Diamond$ is an identity.

2. for the intensional conjunction:

$\mathcal{M} \models_{x,g} \phi \wedge \psi$ iff $\mathcal{M} \models_{x,g} \Box(\phi \wedge_c \psi)$ iff $\forall y (y \leq x \text{ implies } \mathcal{M} \models_{y,g} \phi \wedge_c \psi)$ iff $\forall y (y \leq x \text{ implies } (\mathcal{M} \models_{y,g} \phi \text{ and } \mathcal{M} \models_{y,g} \psi))$ iff $\mathcal{M} \models_{x,g} \phi \text{ and } \mathcal{M} \models_{x,g} \psi$.

Thus, intensional conjunction has the same semantics as extensional conjunction.

3. for the intensional implication:

$\mathcal{M} \models_{x,g} \phi \Rightarrow \psi$ iff $\mathcal{M} \models_{x,g} \Box(\phi \Rightarrow_c \psi)$ iff $\forall y (y \leq x \text{ implies } \mathcal{M} \models_{y,g} \phi \Rightarrow_c \psi)$ iff $\forall y (y \leq x \text{ implies } (\mathcal{M} \models_{y,g} \phi \text{ implies } \mathcal{M} \models_{y,g} \psi))$ iff $\forall y ((y \leq x \text{ and } \mathcal{M} \models_{y,g} \phi) \text{ implies } \mathcal{M} \models_{y,g} \psi)$.

Here $\|(\phi \Rightarrow_c \psi)/g\| = \{y \mid \mathcal{M} \models_{y,g} \phi \text{ implies } \mathcal{M} \models_{y,g} \psi\} \notin \mathcal{F}(X)$, but it holds that $\|\Box(\phi \Rightarrow_c \psi)/g\| = \Gamma \|(\phi \Rightarrow_c \psi)/g\| = \Gamma (\bigcup \{Z \mid Z \cap \|\phi/g\| \subseteq \|\psi/g\|\}) \in \mathcal{F}(X)$.

If $\|\phi/g\|, \|\psi/g\| \in \mathcal{F}(X)$ then $\|\Box(\phi \Rightarrow_c \psi)/g\| = \|\phi/g\| \multimap \|\psi/g\| \in \mathcal{F}(X)$.

4. for the intensional disjunction:

$\mathcal{M} \models_{x,g} \phi \vee \psi$ iff $\mathcal{M} \models_{x,g} \Box(\phi \vee_c \psi)$ iff $\forall y (y \leq x \text{ implies } \mathcal{M} \models_{y,g} \phi \vee_c \psi)$ iff $\forall y (y \leq x \text{ implies } (\mathcal{M} \models_{y,g} \phi \text{ or } \mathcal{M} \models_{y,g} \psi))$ iff $\forall y (y \leq x \text{ implies } (\text{not } \mathcal{M} \models_{y,g} \phi \text{ implies } \mathcal{M} \models_{y,g} \psi))$ iff $\forall y ((y \leq x \text{ and not } \mathcal{M} \models_{y,g} \phi) \text{ implies } \mathcal{M} \models_{y,g} \psi)$.

Here $\|(\phi \vee_c \psi)/g\| = \{y \mid \mathcal{M} \models_{y,g} \phi \text{ or } \mathcal{M} \models_{y,g} \psi\} = \|\phi/g\| \cup \|\psi/g\| \notin \mathcal{F}(X)$, thus $\Box$ is not an identity, but holds that $\|\Box(\phi \vee_c \psi)/g\| = \Gamma\|(\phi \vee_c \psi)/g\| = \Gamma(\|\phi/g\| \cup \|\psi/g\|) = \|\phi/g\| \uplus \|\psi/g\| \in \mathcal{F}(X)$.

Notice that in the case when $(X, \leq)$ is a distributive lattice (when we are using also logic implication and negation defined as in Heyting algebras) and we are using for the set of worlds the set $\widehat{X}$ of join-irreducible elements in X (the second alternative for autoreferential assumption, based on the representation isomorphisms in point 2 of Propositions 15 and 16), we obtain the standard semantics for extensional disjunction, that is, $\mathcal{M} \models_{x,g} \phi \vee \psi$ iff $\mathcal{M} \models_{x,g} \phi$ or $\mathcal{M} \models_{x,g} \psi$.

In this particular case when the set of worlds is the set $\widehat{X} \subset X$, for the logic connectives $\wedge, \vee, \neg, \Rightarrow$ we obtain Kripke-style semantics of intuitionistic logic, but with inverted accessibility relation, of complete distributive lattice, for the operator $\Box$. This version is presented in (Majkić, 2006d) for the complete distributive lattices with additive normal modal operators.

Based on all these considerations we can see how important a role the closure operator and universal "necessity" operator Γ have, based on the Dedekind-McNeile Galois connection over partial order of a complete lattice $(X, \leq)$ of algebraic truth-values.

Thus, the autoreferential semantics based on the lattice preorder, determinate both the Kripke frame for many-valued modal logic and the Kripke-style semantics for logic connectives. The next proposition shows that there is a bijective correspondence (see also in (Majkić, 2006d)) between the *algebraic* many-valued Herbrand models of a modal logic $\mathcal{L}$, based on the complete lattice of truth values $(X, \leq)$, and the *relational* (or *Kripke-style*) models (based on the set of possible worlds equal to the set of truth values X) of the same logic $\mathcal{L}$.

PROPOSITION 27. — (DUALITY INVARIANCE) *For any given Kripke model of a modal logic $\mathcal{L}$, based on a complete lattice $(X, \leq)$ of truth values, which satisfies Definition 26, we are able to obtain the algebraic many-valued Herbrand model $I : H \rightarrow X$ of the same logic $\mathcal{L}$, such that for any ground atom $r(c_1, \dots, c_n) \in H$ holds*
 (a) $I(r(c_1, \dots, c_n)) = \bigvee \{x \in X \mid \mathcal{M} \models_x r(c_1, \dots, c_n)\}$.

Vice versa, for any given algebraic many-valued Herbrand model $I : H \rightarrow X$ of a logic $\mathcal{L}$, we are able to obtain the Kripke model of $\mathcal{L}$, which satisfies Definition 26, such that for any world $x \in X$, a predicate symbol $r \in P$ with arity n , and a tuple of constants $(c_1, \dots, c_n) \in S^n$ holds (b) $V(x, r)(c_1, \dots, c_n) = 1$ iff $x \leq I(r(c_1, \dots, c_n))$.

PROOF. — Easy to verify, directly from Definition 26. The next theorem shows the clear relationships between the canonical *representation algebra* based on the complete lattice $(\mathcal{F}(X), \subseteq)$ and the Kripke style model of a many-valued modal logic $\mathcal{L}$. ■

THEOREM 28. — *Let $K = \langle (X, \leq), \{\mathcal{R}_i\} \rangle$ be a frame and $I : H \rightarrow X$ be a Herbrand valuation of a many-valued logic $\mathcal{L}$, given by Definition 26. Then, for a given assignment g and for any formula ϕ , the set of worlds where ϕ/g (a formula ϕ where all variables are substituted by the assignment g) holds is $\|\phi/g\| = \downarrow (\bar{I}(\phi/g)) \in \mathcal{F}(X)$.*

PROOF. — By structural induction: We have that $x \in \|\phi/g\|$ iff $\mathcal{M} \models_x \phi/g$ iff $\mathcal{M} \models_{x,g} \phi$.

1. For any ground atom $r(c_1, \dots, c_n) \in H$, $x \in X$, $\mathcal{M} \models_{x,g} r(c_1, \dots, c_n)$ iff $x \leq I(r(c_1, \dots, c_n))$, thus $\|r(c_1, \dots, c_n)\| = \downarrow I(r(c_1, \dots, c_n))$. Notice that it can be obtained directly from the equation (a) of Proposition 27, i.e., by applying the operator $\downarrow$ to both sides of this equation (a) we obtain $\downarrow I(r(c_1, \dots, c_n)) = \downarrow \bigvee \{x \in X \mid \mathcal{M} \models_x r(c_1, \dots, c_n)\} = \{x \in X \mid \mathcal{M} \models_x r(c_1, \dots, c_n)\} = \|r(c_1, \dots, c_n)\|$.

Suppose by hypothesis that $\|\phi/g\| = \downarrow (\bar{I}(\phi/g))$ and $\|\psi/g\| = \downarrow (\bar{I}(\psi/g))$, then:

2. From $\mathcal{M} \models_x \phi/g \wedge \psi/g$ iff $\mathcal{M} \models_x \phi/g$ and $\mathcal{M} \models_x \psi/g$, holds that $x \in \|\phi/g \wedge \psi/g\|$ iff $x \in \|\phi/g\| \cap \|\psi/g\|$, (and by structural induction hypothesis) iff $x \in \downarrow \bar{I}(\phi/g) \cap \downarrow \bar{I}(\psi/g) = \downarrow \bar{I}(\phi/g \wedge \psi/g)$. Thus, $\|\phi/g \wedge \psi/g\| = \downarrow \bar{I}(\phi/g \wedge \psi/g)$.
3. From $\mathcal{M} \models_{x,g} \phi \vee \psi$ iff $\forall y((y \leq x \text{ and not } \mathcal{M} \models_{y,g} \phi) \text{ implies } \mathcal{M} \models_{y,g} \psi)$ iff $\forall y(((x \leq y \text{ and not } \mathcal{M} \models_y \phi/g) \text{ implies } \mathcal{M} \models_y \psi/g))$ iff $\forall y((y \leq x \text{ and not } y \leq \bar{I}(\phi/g) \text{ implies } y \leq \bar{I}(\psi/g))$ iff $\forall y((y \in \downarrow x \cap (X - \|\phi/g\|) \text{ implies } y \in \|\psi/g\|))$ iff $\downarrow x \cap (X - \|\phi/g\|) \subseteq \|\psi/g\|$.

So that $S = \|\phi/g \vee \psi/g\| = \{x \mid \downarrow x \cap (X - \|\phi/g\|) \subseteq \|\psi/g\|\}$.

Then, $S = id_{\mathcal{F}(X)}(S) = \downarrow \bigvee S = \biguplus \downarrow S$ (from the homomorphism $\downarrow$) = $\biguplus_{x \in S} \downarrow x = \biguplus \{\downarrow x \mid \downarrow x \cap (X - \|\phi/g\|) \subseteq \|\psi/g\|\} = \downarrow \bigvee \{\downarrow x \mid \downarrow x \cap (X - \|\phi/g\|) \subseteq \|\psi/g\|\} =$ (from (1) in Proposition 7) $= \downarrow \bigvee ((X - (X - \|\phi/g\|)) \cup \|\psi/g\|) = \downarrow \bigvee (\|\phi/g\| \cup \|\psi/g\|) =$ (from inductive hypothesis) $= \downarrow \bigvee (\downarrow \bar{I}(\phi/g) \cup \downarrow \bar{I}(\psi/g)) =$ (from the homomorphism $\downarrow$) $= \downarrow \bigvee (\bar{I}(\phi/g) \vee \bar{I}(\psi/g)) = \downarrow (\bar{I}(\phi/g) \vee \bar{I}(\psi/g))$.

That is, $\|\phi/g \vee \psi/g\| = \downarrow \bar{I}(\phi/g \vee \psi/g)$.

The proof for the alternative definition 3.a is as follows: From $\mathcal{M} \models_{x,g} \phi \vee \psi$ iff $\exists y, z(x \leq y \vee z \text{ and } \mathcal{M} \models_{y,g} \phi \text{ and } \mathcal{M} \models_{z,g} \psi)$, holds that $x \in \|\phi/g \vee \psi/g\|$ iff (by structural induction hypothesis) $\exists y, z(x \leq y \vee z \text{ and } y \leq \bar{I}(\phi/g) \text{ and } z \leq \bar{I}(\psi/g))$ iff $x \leq \bigvee \{y \vee z \mid y \leq \bar{I}(\phi/g) \text{ and } z \leq \bar{I}(\psi/g)\} = \bar{I}(\phi/g) \vee \bar{I}(\psi/g) = \bar{I}(\phi/g \vee \psi/g)$. Thus, $\|\phi/g \vee \psi/g\| = \downarrow \bar{I}(\phi/g \vee \psi/g)$.

4. From $x \in \|\phi/g \Rightarrow \psi/g\|$ iff $\mathcal{M} \models_x \phi/g \Rightarrow \psi/g$ iff $\forall y(((x \leq y \text{ and } \mathcal{M} \models_y \phi/g) \text{ implies } \mathcal{M} \models_y \psi/g))$ iff $\forall y((y \leq x \text{ and } y \leq \bar{I}(\phi/g))$

implies $y \leq \bar{I}(\psi/g)$ iff $\forall y (y \in \downarrow x \cap \|\phi/g\| \text{ implies } y \in \|\psi/g\|)$ iff $\downarrow x \cap \|\phi/g\| \subseteq \|\psi/g\|$. So that $S = \|\phi/g \Rightarrow \psi/g\| = \{x \mid \downarrow x \cap \|\phi/g\| \subseteq \|\psi/g\|\}$.

Then, $S = id_{\mathcal{F}(X)}(S) = \downarrow \bigvee S = \bigcup \downarrow S$ (from the homomorphism $\downarrow$) = $\bigcup_{x \in S} \downarrow x = \bigcup \{\downarrow x \mid \downarrow x \cap \|\phi/g\| \subseteq \|\psi/g\|\} = \bigcup \{S' \in \mathcal{F}(X) \mid S' \cap \downarrow \bar{I}(\phi/g) \subseteq \downarrow \bar{I}(\psi/g)\} = \downarrow \bar{I}(\phi/g) \rightarrow \downarrow \bar{I}(\psi/g) = \downarrow (\bar{I}(\phi/g) \rightarrow \bar{I}(\psi/g)) = \downarrow \bar{I}(\phi/g \Rightarrow \psi/g)$ (from the homomorphism of the valuation $\bar{I}$). Consequently, $\|\phi/g \Rightarrow \psi/g\| = \downarrow \bar{I}(\phi/g \Rightarrow \psi/g)$.

5. For any additive algebraic modal operator o_i we obtain the existential logic modal operator $\Diamond_i$, so that $\mathcal{M} \models_x \Diamond_i \phi/g$ iff $\exists y \in X((x, y) \in \mathcal{R}_i \text{ and } \mathcal{M} \models_y \phi/g)$, iff $\exists y((x, y) \in \mathcal{R}_i \text{ and } y \leq \bar{I}(\phi/g))$.

Let us denote it by $\alpha = \bar{I}(\phi/g)$, then the condition above, $(*) \exists y((x, y) \in \mathcal{R}_i \text{ and } y \leq \alpha)$, is satisfied in one of the following possible cases:

5.1.1 Case when $x \in \pi_1(S_m)$. Then $(*)$ corresponds to $x = o_i(y)$ and $o_i(y) \leq o_i(\alpha)$, (because o_i is monotone), that is, $x \leq o_i(\alpha)$, i.e., $x \in \downarrow o_i(\alpha)$;

5.1.2 Case when $x \notin \pi_1(S_m)$, but $(x, y) \in \mathcal{R}_i$. Then, $\exists(v, y) \in S_m.(x \leq v)$ i.e., $x \leq v = o_i(y)$, and from $(*)$, $y \leq \alpha$, and from the monotonicity of o_i , $o_i(y) \leq o_i(\alpha)$, we obtain $x \leq o_i(\alpha)$, i.e., $x \in \downarrow o_i(\alpha)$.

Vice versa, let us show that for any $x \in \downarrow o_i(\alpha)$, the condition $(*)$ is satisfied:

5.2.1 Case when $x = o_i(\alpha)$. We have $(o_i(\alpha), \alpha) \in S_m \subseteq \mathcal{R}_i$, thus, $y = \alpha$ and $(*)$ is satisfied;

5.2.2 Case when $x < o_i(\alpha)$ and $\exists y.x = o_i(y)$. Then $(x, y) \in S_m \subseteq \mathcal{R}_i$. Suppose that $y > \alpha$, then from the monotonicity of o_i it will hold that $o_i(y) > o_i(\alpha) > x$, which is in contrast with $x = o_i(y)$. So, it must hold that $y \leq \alpha$, and $(*)$ is satisfied;

5.2.3 Case when $x < o_i(\alpha)$ and $\nexists y.x = o_i(y)$. Then $(z, y) \in S_m \subseteq \mathcal{R}_i$ such that $z = o_i(y) \geq x$, and consequently $o_i(\alpha) \in \{o_i(y_1) \mid (x, y_1) \in \mathcal{R}_i \setminus S_m\}$, and $(x, y) \in \mathcal{R}_i \setminus S_m$ for $y = \alpha$ (from the fact that $x \leq o_i(y) = o_i(\alpha)$), so that $(*)$ is satisfied.

Thus, we have $\|\Diamond_i \phi/g\| = \downarrow o_i(\alpha) = \downarrow o_i(\bar{I}(\phi/g)) = \downarrow \bar{I}(\Diamond_i \phi/g)$.

6. For any negation operator $\tilde{o}_i$ we obtain the modal operator $\sim_i$, we have $\mathcal{M} \models_x \sim_i \phi$ iff $\forall y(\mathcal{M} \models_y \phi \text{ implies } (y, x) \in \widetilde{\mathcal{R}_i})$ iff $\forall y(y \in \|\phi\| \text{ implies } (y, x) \in \widetilde{\mathcal{R}_i})$ iff (by inductive hypothesis $\|\phi\| = \downarrow \alpha$ where $\alpha = \bar{I}(\phi/g)$) $\forall y(y \leq \alpha \text{ implies } (y, x) \in \widetilde{\mathcal{R}_i})$, i.e., $\|\sim_i \phi\| = \{x \mid \forall y(y \leq \alpha \text{ implies } (y, x) \in \widetilde{\mathcal{R}_i})\}$.

It is easy to verify that for each $x \leq \neg\alpha$ and $y \leq \alpha$, $(y, x) \in \widetilde{\mathcal{R}_i}$ (from Definition 23 and the point 2 of Lemma 24), we have that $x \in \|\sim_i \phi\|$. Thus $\|\sim_i \phi\| \supseteq \downarrow \alpha$.

Suppose that there exists $x > \neg\alpha$ such that $x \in \|\sim_i \phi\|$. In that case there are two possible cases (here S is the binary relation in Lemma 24 of this negation operator):

6.1 There exists $\beta \in \pi_1 S \subseteq \pi_1 \widetilde{\mathcal{R}}_i$ such that $x = \neg\beta > \neg\alpha$: then (from the point 1 of Lemma 24) $\beta < \alpha$, and $(\beta, \neg\beta) \in S \subseteq \widetilde{\mathcal{R}}_i$, but $(\alpha, \neg\beta) \notin \widetilde{\mathcal{R}}_i$ (from Definition 23) so that $\forall y (y \leq \alpha \text{ implies } (y, \neg\beta) \in \widetilde{\mathcal{R}}_i)$ is false (the case when $y = \alpha$ cannot be satisfied), in contradiction with the hypothesis.

6.2 Does not exist β such that $x = \neg\beta > \neg\alpha$: then $(\alpha, x) \in \widetilde{\mathcal{R}}_i$ only if $\exists \alpha_1. \neg\alpha_1 > x$ with $(\alpha_1, \neg\alpha_1) \in S \subseteq \widetilde{\mathcal{R}}_i$ (from Definition 23), but from $\neg\alpha_1 > x > \neg\alpha$ we have (from the point 1 of Lemma 24) that $\alpha_1 < \alpha$, so that $(\alpha, \alpha_1) \notin \widetilde{\mathcal{R}}_i$, and, consequently, $(\alpha, x) \notin \widetilde{\mathcal{R}}_i$. Thus $\forall y (y \leq \alpha \text{ implies } (y, x) \in \widetilde{\mathcal{R}}_i)$ is false, in contradiction with hypothesis. Consequently, $\|\sim_i \phi\| = \downarrow \alpha$. ■

REMARK. — Notice that in the proof for existential modal operators, we used only the *monotonic* property for these operators and not its additive property. That means that the definition of the satisfaction for existential modal operators (point 6 in Definition 26) can be extended to *any monotonic operator*, and, particularly to *universal* modal operators also. In this case, we are able to have *the same* existential definition for both joined modal operators, but with each one based on its proper specific accessibility relation.

This is important if we consider that the satisfaction for universal modal operators $\mathcal{M} \models_x \Box_i \phi/g$ is not given by the classic definition, that is by $\forall y \in X ((x, y) \in \mathcal{R}_{\Diamond_i} \text{ implies } \mathcal{M} \models_y \phi/g)$, where $\mathcal{R}_{\Diamond_i}$ is the accessibility relation given by Definition 21 for the existential modal operator. It can not be defined by $\mathcal{M} \models_x \neg\Diamond_i \neg\phi/g$ because, in distributive lattices, for two adjoint modal operators $\Diamond_i \dashv \Box_i$ generally does not hold $\Diamond_i = \neg\Box_i \neg$ and $\Box_i = \neg\Diamond_i \neg$.

Thus, we can define the semantics for the satisfaction of universal modal operator by this new way, that is

7.a $\mathcal{M} \models_x \Box_i \phi/g$ iff $\exists y \in X ((x, y) \in \mathcal{R}_{\Box_i} \text{ and } \mathcal{M} \models_y \phi/g)$, where $\mathcal{R}_{\Box_i}$ is an accessibility relation obtained by Definition 21 applied to the universal modal operator $\Box_i$, such that $\mathcal{R}_{\Box_i} \neq \mathcal{R}_{\Diamond_i}$, or by dual result, when $\Diamond_i$ is an additive modal operator, obtained for accessibility relations of adjoint unary modal operators and for the "gaggle" theory of Dunn (here $\overline{A}$ denotes the set complement of A)

7.b $\mathcal{M} \models_x \Box_i \phi/g$ iff $\forall y \in X ((x, y) \in (\overline{\mathcal{R}_{\Diamond_i}})^{-1} \text{ implies } \mathcal{M} \models_y \phi/g)$.

This is really a new result which generalizes the relational semantics for *normal* Kripke modal logic: for any couple of adjoint modal operators $l \vdash r$ (Galois connection), which results in a normal Kripke modal logic, we are able to define the satisfaction relation of these two (universal versus existential) operators based on two accessibility relations, but in the same standard way used only for existential modal operators.

This result is particularly important when we are dealing with *positive modal logic* (Celani *et al.*, 1997; Dunn, 1995) (without negation) for which Definition 21 holds also if we substitute point 7, with this new definition for universal modal operator, based on its proper accessibility relation which is different from the relation used for an existential modal operator. $\square$

EXAMPLE 29. — In the case of Belnap's 4-valued logic, where the logic values can be considered as 0-arity atoms, we can verify that $\mathcal{M} \models_{\perp, g} \neg_t \top$ iff $\mathcal{M} \models_{\perp, g} (\top \Rightarrow \emptyset)$ iff $\forall y((y \leq \perp \text{ and } \mathcal{M} \models_{y, g} \top) \text{ implies } \mathcal{M} \models_{y, g} \emptyset)$ which is true (consider that $\mathcal{M} \models_{y, g} \emptyset$ is true only if $y = f$, and false otherwise), thus $\|\neg_t \top\| = \downarrow I(\neg_t \top) = \downarrow I(\perp) = \downarrow \perp = \{f, \perp\}$.

While $\mathcal{M} \models_{\top, g} \neg_t \top$ iff $\mathcal{M} \models_{\top, g} \top \Rightarrow \emptyset$ iff $\forall y((y \leq \top \text{ and } \mathcal{M} \models_{y, g} \top) \text{ implies } \mathcal{M} \models_{y, g} \emptyset)$ is false (consider the case when $y = \top$, then $y \leq \top$ and $\mathcal{M} \models_{y, g} \top$ is true, while $\mathcal{M} \models_{\top, g} \emptyset$ is false), consequently $\top \notin \|\neg_t \top\|$.

The Kripke frame for autoepistemic intuitionistic logic based on Belnap's 4-valued logic is very simple, with two accessibility relations: $\mathcal{R}_- = \{(f, f), (\top, \perp), (\perp, \top), (t, t)\}$, for the belief (conflation) modal operator, and $\mathcal{R}_\mu = \{(f, f), (t, \top), (\top, \top), (\top, t), (\perp, \top), (\perp, t)\}$, for the Moore's autoepistemic modal operator μ .

Both of them can be obtained from Definition 21 by applying it independently to the existential or universal operator, just because in these two cases they are selfadjoint (equal).

Let us consider a more general case when the existential and universal operator are different, given by the following truth table:

	M	L
t	t	$\top$
$\top$	t	$\top$
$\perp$	$\perp$	f
f	$\perp$	f

It is easy to verify that the operator M is the multiplicative monotone operator (*i.e.*, $M(\alpha \wedge \beta) = M(\alpha) \wedge M(\beta)$ and $M(t) = t$), used for universal modal logic operator $\square$. The operator L is its correspondent (by Galois connection $L \dashv M$) additive operator (*i.e.*, $L(\alpha \vee \beta) = L(\alpha) \vee L(\beta)$ and $L(f) = f$), used for an existential modal logic operator $\diamond$, such that $M = \neg_t L \neg_t$ (because the Belnap's lattice $\mathcal{B}_4$ is a Boolean algebra) with $\neg_t \neg_t = id$ an identity.

The accessibility relations for these two monotone operators, obtained from Definition 21 are, $\mathcal{R}_\diamond = \{(\top, \top), (f, f)\}$ and $\mathcal{R}_\square = \{(t, \top), (\top, \top), (\top, t), (\perp, f), (f, f)\}$, and we will denominate them as *joint accessibility relations*.

Thus, for the existential modal operator the satisfaction is as usual $\mathcal{M} \models_x \diamond \phi / g$ iff $\exists y \in X((x, y) \in \mathcal{R}_\diamond \text{ and } \mathcal{M} \models_y \phi / g)$, while for the universal modal operator

we have two alternative definitions $\mathcal{M} \models_x \Box\phi/g$ iff $\mathcal{M} \models_x \neg\Diamond\neg\phi/g$ iff $\forall y \in X((x, y) \in \mathcal{R}_\Diamond \text{ implies } \mathcal{M} \models_y \phi/g)$, based on the relation $\mathcal{R}_\Diamond$, or, independently, based on accessibility relation $\mathcal{R}_\Box$, $\mathcal{M} \models_x \Box\phi/g$ iff $\exists y \in X((x, y) \in \mathcal{R}_\Box \text{ and } \mathcal{M} \models_y \phi/g)$. $\square$

Finally, from the canonical representation for a complete distributive lattice based modal intuitionistic logics in Proposition 16, we obtained that the isomorphism, between the original Galois algebra $G = (X, \leq, \wedge, \vee, \neg, \rightarrow, \{o_i\}_{i \leq \omega}, 0, 1)$ with unary modal operators in $\{o_i\}_{i \leq \omega}$ and its *canonical* representation algebra $\mathbf{G}_K = (\mathcal{F}(X), \subseteq, \cap, \cup, \hat{\neg}, \hat{\rightarrow}, \{\hat{o}_i\}_{i \leq \omega}, 0, 1)$, where $\hat{o}_i = \downarrow o_i \downarrow^{-1}$, $i = 1, 2, \dots$, corresponds to the representation of any ground formula ϕ/g by the set of worlds $\|\phi/g\|$ in the canonical Kripke model for the modal algebra G .

So, for example, a ground term $\phi/g \wedge \psi/g$ in a predicate modal logic $\mathcal{L}$ (based on the Galois algebra G) with a many-valued Herbrand model $I : H \rightarrow X$, corresponds to the set $\|\phi/g\| \cap \|\psi/g\|$ in the canonical algebra $\mathbf{G}_K \simeq G$, where $\|\phi/g\| = \downarrow y$ (y is an algebraic truth-value in a Herbrand model I for the ground formula ϕ/g) is equal to the set of worlds in the canonical Kripke model $\mathcal{M} = (K, S, V)$ (given by Definition 26) where ϕ/g holds.

That is the result of the *autoreferential* representation, where the set of possible worlds is assumed to be the set of algebraic truth values (the complete lattice), used for the canonical and Kripke style semantics for modal many-valued logics.

6. Autoreference, manyvaluedness and paraconsistency

I will begin this section with some words (Cadoli *et al.*, 1996) from my friend from Roma, Marco Cadoli (1965-11/21/2006) recently deceased:

“The main motivation for adopting MV logic was the necessity to overcome some semantic limitations of classical logic. One of the drawbacks of classical logic that has been more frequently pointed out is that a generic unsatisfiable formula —*e.g.* $a \wedge \neg a$ — implies *any* formula b . The work of Anderson and Belnap (Anderson *et al.*, 1975) in relevance logic has been one of the first attempts at defining a semantically well-founded logical system that does not have this undesirable feature. As shown by Belnap in (Belnap, 1977) their system can indeed be characterized by a multivalued semantics.”

The *paraconsistent* logic goes beyond consistency but prevents triviality, *i.e.*, the explosive inconsistency of classic logic described above. In paraconsistent logics we can have a kind of local inconsistency which has no total explosive feature as in 2-valued classical logic. Thus, in the presence of these local inconsistencies paraconsistent logic still separates propositions into two non-empty classes, derivable and non-derivable ones.

The most important criteria for choosing paraconsistent logics, as suggested in (Nelson, 1959; da Costa, 1974; Majkić, 2008), and recently formalized into the Logic of Formal Inconsistencies (LFI) (Carnielli *et al.*, 2006), is their abstract degree of non-triviality, rather than the mere absence of contradiction.

The LFI internalizes the very notions of consistency and inconsistency at the object-language level, instead of the usual meta-logic level. It designs an expressive logic language, able of recovering both consistent reasoning and allowing for some inconsistency.

We will denote by $\mathcal{L}$ a many-valued predicate logic, with a Herbrand base H (the set of all ground atoms), based on a complete distributive lattice $(X, \leq)$ of algebraic truth-values, obtained as free algebra by a carrier set H of ground atoms and the set of logic connectives in Σ , that is, the set of standard logic connectives $\{\wedge, \vee, \Rightarrow, \neg\}$, corresponding to the Heyting algebra in Proposition 15, and the *finite* number of unary modal (normal and *non* normal) operators. We will denote by A_X the extended many-valued Heyting algebra by a finite set of modal operators $\{o_i : X \rightarrow X\}_{i < \omega}$, that is, $A_X = (X, \leq, \wedge, \vee, \Rightarrow, \neg, \{o_i : X \rightarrow X\}_{i < \omega})$.

Any many-valued valuation $v : H \rightarrow X$, *i.e.*, Herbrand interpretation, can be uniquely extended to all formulae in $\mathcal{L}_G$ (the set of all ground formulae in $\mathcal{L}$) by the homomorphism $\bar{v} : \mathcal{L}_G \rightarrow A_X$. By (A_X, D) we will denote the matrix where $D \subseteq X$ is any subset of designated algebraic values.

We will formally define the multi-modal many-valued logic as a pair $(\mathcal{L}_G, \Vdash)$ where $\Vdash \subseteq \mathcal{P}(\mathcal{L}_G) \times \mathcal{L}_G$ is a *consequence relation* for a logic $\mathcal{L}_G$, generally defined as follows: for any *theory* Δ (a subset of ground formulae of $\mathcal{L}_G$), the ground formula $\phi \in \mathcal{L}_G$ is a logical consequence of Δ , denoted by $\Delta \Vdash \phi$, iff $\forall v (\forall \psi \in \Delta. \bar{v}(\psi) \in D \text{ implies } \bar{v}(\phi) \in D)$.

It is said that a theory Δ is *closed* iff $\{\phi \in \mathcal{L}_G \mid \Delta \Vdash \phi\} = \Delta$.

In what follows we will briefly present basic concepts in LFI, and invite the readers to use (Carnielli *et al.*, 2002; Carnielli *et al.*, 2006) for more information. We will omit 'ground' when we use the word formula; notice that each ground atom can be seen as a propositional variable, so that grounded predicate logic can be equivalently considered as a propositional logic with the set of propositional variables isomorphic to the Herbrand base of this predicate logic.

The logic $(\mathcal{L}_G, \Vdash)$ with a 'negation' $\sim$, which has to be antitonic and must satisfy $\sim 1 = 0$, and $\sim 0 = 1$, but different from strong *supplementary negation* (Carnielli *et al.*, 2006) based on pseudocomplement, $\neg\alpha = \alpha \multimap 0$, where $\multimap$, defined as relative pseudocomplement, is a *deductive implication* (Carnielli *et al.*, 2006) which satisfies the Modus Ponens and Deduction Theorem (Majkić, 2006b) (notice that w.r.t (Carnielli *et al.*, 2006) here we inverted the symbols for these two negations), is:

- 1) contradictory, if $\forall \Delta \exists \phi (\Delta \Vdash \phi, \text{ and } \Delta \Vdash \sim \phi)$.
- 2) trivial, if $\forall \Delta \forall \phi (\Delta \Vdash \phi)$.
- 3) explosive, if $\forall \Delta \forall \phi \forall \psi (\Delta, \phi, \sim \phi \Vdash \psi)$.

4) *gently explosive*, if $\forall \Delta \forall \phi \forall \psi (\Delta, \phi, \sim \phi, O(\phi) \Vdash \psi)$,

where $O(\phi)$ is a possibly empty set of formulae which depends only on ϕ , satisfying the following: there are formulae α, β such that $O(\alpha), \alpha \nVdash \beta$ and $O(\alpha), \sim \alpha \nVdash \beta$.

5) *consistent*, if it is both explosive and non-trivial.

6) *paraconsistent*, if it is inconsistent yet not trivial.

In (Carnielli *et al.*, 2006) it was shown that for tarskian (monotonic) logic hold:

1) if logic is trivial then it is both contradictory and explosive,

2) an explosive logic fails non-triviality iff it fails non-contradiction.

Finally, in (Carnielli *et al.*, 2006) a Logic of Formal Inconsistency (LFI) is defined as a logic which is not explosive but is *gently explosive*. Obviously, LFI represents a very significant part of the class of paraconsistent logics where $\sim$ is called *paraconsistent negation*.

Differently from LFI, we consider all kinds of many-valued logics (also non monotonic where the property $(\Delta \Vdash \phi \text{ and } \Delta \subseteq \Lambda) \text{ implies } \Lambda \Vdash \phi$) based on a complete distributive lattice of algebraic truth values does not hold.

In what follows we will demonstrate that all Many-Valued logics based on a Complete *distributive* Lattice of truth-values (MVCL logics) are gently explosive, that is they are LFIs.

Now we will define the paraconsistent negation $\sim$ for MVCL logics. Given an unary operator $\odot : X \rightarrow X$ we will denote its image by $im\odot = \{y = \odot(x) \mid x \in X\}$. For a paraconsistent negation we need the condition $\exists x \in X. (x \wedge \sim x \neq 0)$ (in that case we take that $x \wedge \neg x \in D$ is a designated element), so it must hold $\mathbf{2} \subset im\sim$.

PROPOSITION 30. — *Strong negation $\neg$, defined by the pseudocomplement $\neg\alpha = \alpha \rightarrow 0$ in a complete distributive lattice $(X, \leq)$, can not be used as a paraconsistent negation.*

PROOF. — In fact, for any $x \in X$ we have that $x \wedge \neg x = x \wedge (\bigvee \{y \mid x \wedge y = 0\}) = \bigvee \{x \wedge y \mid x \wedge y = 0\} = 0$. Thus, $\nexists x \in X. x \wedge \neg x \neq 0$, and $\neg$ is the supplementary negation. ■

An *atom* in a lattice $(X, \leq)$ is an element $x \in X$ different from 0, such that $\nexists y \neq 0. (y < x)$. Now we will show how we are able to define the paraconsistent negation for any complete distributive lattice $(X, \leq)$.

PROPOSITION 31. — *Any bounded lattice $(X, \leq)$, with cardinality bigger than 2, can have a paraconsistent negation $\sim$.*

PROOF. — For any fixed $\alpha \in X$, such that $\alpha \notin \{0, 1\}$, we define

$\sim x = 1$ if $x = 0$; 0 if $x = 1$; α if $x = \alpha$. It is easy to show that for any two $x, y \in X$, if $x \leq y$ then $\neg x \geq \neg y$. So that $\sim$ is a general negation given by Definition 11. It is easy to verify that $\alpha \wedge \neg\alpha = \alpha \neq 0$.

But in some cases there are also other kinds of paraconsistent negation:

1) When X is an infinite set and there exists the isomorphism (which preserves the ordering) with the closed interval of reals $[a, b]$, $\omega : (X, \leq, \wedge, \vee) \simeq ([a, b], \leq, \min, \max)$, then we define for any $x \in X$, $\sim x = \omega^{-1}(b + a - \omega(x))$, where ω^{-1} is the inverse of ω .

2) When X is finite with the cardinality $n = |X| \geq 3$. We define the following isomorphism $\sigma : (X, \leq, \wedge, \vee) \simeq (\{\frac{i}{n} \mid 1 \leq i \leq n\}, \leq, \min, \max)$, and we define for any $x \in X$, $\sim x = \sigma^{-1}(\frac{1+n}{n} - \sigma(x))$, where σ^{-1} is the inverse of σ . ■

EXAMPLE 32. — Let us consider the following cases:

1) In the case of fuzzy logic, where the complete lattice is a closed interval of reals, $X = [0, 1]$, we have the case 1.1 where ω is defined by $\omega(x) = a + (b - a) * x$ for any $x \in X$, and $\omega^{-1}(y) = (y - a)/(b - a)$ for any $y \in [a, b]$. Thus $\sim x = \omega^{-1}(b + a - \omega(x)) = (b + a - \omega(x) - a)/(b - a) = 1 - (\omega(x) - a)/(b - a) = 1 - \omega^{-1}(\omega(x)) = 1 - x$ is a standard negation of fuzzy logic.

The set of designated elements is $D = \{y \mid y \in X \text{ and } a \leq y\}$ for some $0 < a \leq 1$.

2) In the case of the 3-valued logic where $X = \{0, \frac{1}{2}, 1\}$, and $\sigma(x) = \frac{1+2x}{3}$, we obtain $\sim(0) = 1$, $\sim(\frac{1}{2}) = \frac{1}{2}$ and $\sim(1) = 0$.

The set of designated elements is $D = \{\frac{1}{2}, 1\}$.

3) In the case of Belnap's 4-valued logic, $X = \{f, \top, \perp, t\}$, where 0 corresponds to f and 1 to t , we have two atoms, $\top$ and $\perp$, and we obtain that $\sim = \neg_t -$ is standard epistemic bilattice negation, where $\neg_t$ is a pseudocomplement (supplementary negation) and $-$ the belief modal operator (conflation).

The set of designated elements is $D = \{\top, t\}$.

4) As one example we can consider the following *explicit paraconsistent* extension of fuzzy logic, where $X = [0, .5) \cup \{.5^-, .5^+\} \cup (.5, 1]$, where $.5^-$, $.5^+$ substitute the "normal" value $.5$, but only with the property that $.5^- \bowtie .5^+$. So that X is not a total ordering. In this case the number of atoms is zero. We can consider $.5^+$ as an "inconsistent" value (like $\top$ in Belnap's bilattice), and $.5^-$ as an "unknown" value (like $\perp$ in Belnap's bilattice). We define the negation by $\sim x = x$ if $x \in \{.5^-, .5^+\}$; $1 - x$ otherwise. The set of designated elements is $D = \{.5^-, .5^+\} \cup (.5, 1]$. □

Now we will introduce the particular non normal modal operators (they are not monotone functions), used in (Majkić, 2006e) for the reduction of any MVCL logic into 2-valued modal logic, and, based on them, the paraconsistent negation $\sim$. Then we will define the consistency operator $\circ : X \rightarrow X$ as follows:

DEFINITION 33. — Given a complete lattice $(X, \leq)$ we define the family of unary modal operators $[x] : X \rightarrow X$, for each $x \in X$, as follows: for any $\alpha \in X$ $[x]\alpha = 1$ if $\alpha = x$; 0 otherwise.

Based on them we define the following global modal CONSISTENCY operator $\circ : X \rightarrow X$ for a lattice $(X, \leq)$: for any $\alpha \in X$ $\circ \alpha =_{def} \bigwedge \{\sim [x]\alpha \mid x \in X \setminus 2\}$, where $\setminus$ is the set subtraction.

Notice that in the case of the classic 2-valued logic we obtain for any formula ϕ , $\circ \phi = 1$, because the meet of the empty set is equal to 1.

Otherwise, for a n -valued logic, $\circ \alpha = 1$ for $\alpha \in \mathbf{2}$ and is otherwise equal to 0.

THEOREM 34 (MANYVALUEDNESS AND PARAConsISTENCY). — *Any many-valued logic based on a complete lattice $(X, \leq)$ with $|X| \geq 3$ is a LFI with the consistency operator defined in Definition 33.*

PROOF. — It is easy to verify that the following matrix holds:

	$\sim$	$[x_1]$	$[x_2]$		$[x_i]$		$[x_n]$	$\circ$
1	0	0	0		0		0	1
x_1	y_1	1	0		0		0	0
x_2	y_2	0	1		0		0	0
$\cdot$	$\cdot$	$\cdot$	$\cdot$		$\cdot$		$\cdot$	$\cdot$
$\cdot$	$\cdot$	$\cdot$	$\cdot$		$\cdot$		$\cdot$	$\cdot$
x_i	y_i	0	0		1		0	0
$\cdot$	$\cdot$	$\cdot$	$\cdot$		$\cdot$		$\cdot$	$\cdot$
$\cdot$	$\cdot$	$\cdot$	$\cdot$		$\cdot$		$\cdot$	$\cdot$
x_n	y_n	0	0		0		1	0
0	1	0	0		0		0	1

where for all $1 \leq i \leq n$, $x_i \in X \setminus \mathbf{2}$ and $y_i \in X \setminus \mathbf{2}$, and $n \geq 1$.

It is easy to verify that $\forall x \in X. (x \wedge \sim x \wedge \circ x = 0)$, consequently $\forall \Delta \forall \phi \forall \psi (\Delta, \phi, \sim \phi, O(\phi) \Vdash \psi)$, while $\circ 1, 1 \nVdash 0$ and $\circ 0, \sim 0 \nVdash 0$.

Thus, a many-valued logic based on a complete lattice $(X, \leq)$ of algebraic truth-values is gently explosive, *i.e.*, it is a LFI. ■

Consequently, for any finite many-valued logic (each finite lattice is a complete lattice) we are able to define the paraconsistent negation in Proposition 31 in order to obtain a LFI. It is possible also in the case of infinite set of algebraic truth-values, when this set is a complete lattice (as, for example, in the case of the fuzzy logic in Example 20).

The relationship between the many-valuedness and the autoreferential assumption is given in precedent sections by representation theorems and dual Kripke-style semantics for many-valued logics based on complete lattices of truth-values (Proposition 16 and Definitions 21,26).

All that remains is to explain the last relationship in the triangle manyvaluedness-autoreference-paraconsistency: the relationship between the *autoreference* assumption for Kripke-style semantics of many-valued logics and *paraconsistency*.

Let us consider the multi-modal many-valued logic $(\mathcal{L}, \Vdash)$ based on the extended many-valued Heyting algebra $A_X = (X, \leq, \wedge, \vee, \multimap, \neg, \{o_i : X \rightarrow X\}_{i < \omega})$, and a particular set of formulae (theory) $\Delta \subset \mathcal{L}$, with the Herbrand base H , for which the Kripke style semantics is given by Definitions 21 and 26.

Let $I : H \rightarrow X$ be a many-valued Herbrand model of this theory Δ , and $\bar{I} : \mathcal{L}_G \rightarrow A_X$ its unique homomorphic extension to all ground formulae. Then from the Theorem 28 we have that the set of ground formulae, which are satisfied in a world $x \in X$ is: $S(x) =_{def} \{\phi \in \mathcal{L}_G \mid \bar{I}(\phi) \geq x\}$.

If we consider a world x as the matrix $(A_X, D(x))$ where $D(x) = \{y \in X \mid y \geq x\}$ is the set of designated elements of this matrix, then the set $S(x)$ is the set of formulae which are satisfied w.r.t this matrix. That is $\forall \phi \in S(x) (\bar{I}(\phi) \in D(x))$, so that I is a model for all formulae in $S(x)$.

Thus we can introduce the following preorder between matrices : $(A_X, D(x)) \preceq (A_X, D(y))$ iff $S(x) \supseteq S(y)$ iff $x \leq y$.

Consequently, the bottom matrix is $(A_X, D(0)) = (A_X, X)$. It corresponds to the bottom world 0, where the set $S(0) = \mathcal{L}_G$ is the set of all ground formulae: this world is *trivial*, and corresponds to classic explosive inconsistency. So that this world can not be paraconsistent.

The top world 1 corresponds to the matrix $(A_X, D(1)) = (A_X, \{1\})$, with the smallest set of formulae $S(1)$. In this world we can not have any true formula $\phi \wedge \neg\phi$, because if $\bar{I}(\phi) \in D(1) = \{1\}$, that is, $\bar{I}(\phi) = 1$, then $\bar{I}(\neg\phi) = 0$, i.e., $\bar{I}(\neg\phi) \notin D(1) = \{1\}$, and, consequently $\bar{I}(\phi \wedge \neg\phi) \notin D(1) = \{1\}$. That is, also this world can not be paraconsistent.

Thus, to be able to permit paraconsistent reasoning we need at least three different worlds, that is n-valued logic with $n \geq 3$.

In that case, for the world $0 < x < 1$ we are able to have the formulae $\phi \wedge \neg\phi$ such that $\bar{I}(\phi \wedge \neg\phi) = x$, that is, $\bar{I}(\phi \wedge \neg\phi) \in D(x)$. Consequently, this world is paraconsistent.

An interesting point is how the hierarchy between the possible worlds (lattice ordering of truth-values) corresponds to the hierarchy of matrices for the same theory, and, consequently, to the hierarchy of paraconsistent logics. The level of paraconsistency is inverse to the truth-level of the possible world: the higher truth level of the possible world corresponds to a lower paraconsistent logic (with a smaller number of paraconsistent formulae).

7. Conclusion

The main result of this paper is a general way of establishing links between algebraic and Kripke-style semantics for the class of many-valued modal logics based on a complete lattice of truth values, by introducing autoreferential duality: it means that we will use as a set of possible worlds in Kripke-style semantics for these modal logics exactly the lattice of its truth values.

This autoreferential approach, where a Kripke frame is based on posets (the complete lattices of truth values), is the result of the consideration that each possible world

represents a level of *credibility*, so that only the propositions with the right logic value (*i.e.*, level of credibility) can be accepted by this world. The lowest credible world 0 is the trivial world of explosive inconsistency: so any derivable formula in this world is strongly doubted. The top world 1 has maximal credibility: it corresponds to classic logic where we have no doubt about true formulae. The intermediate worlds have a degree of credibility, and we can consider true also formulae which in classic logic would be inconsistent: these worlds permit paraconsistent reasoning.

We developed a general Representation Theorem for the class of logics based on complete lattices and on the Dedekind-McNeile Galois connection for the partial order relation of the lattice, with the derived closure operator. The obtained set-based isomorphic algebra has the usual set intersection of a meet operator, but the join operator is a generalization of the set union: we introduced also the subset of complete lattices, that is, the distributive complete lattices, where join corresponds exactly to set union.

We have shown that each distributive complete lattice naturally supports intuitionistic implication and negation, and a set of modal operators, so that it can be extended into an Galois algebra. We defined the Kripke frame and Kripke models for this class of many-valued modal algebraic logics, and have shown that each Kripke model, for such modal logics, is equivalent to the standard algebraic semantics based on many-valued Herbrand models. Differently from the standard method based on the *natural duality theorem* (Clark *et al.*, 1998), where a class of relational structures would be the family of duals of algebras, which is difficult to describe in a simple logic language, our approach offers a very simple and compact description. We demonstrated a general way to obtain paraconsistent negation and consistency operators for this class of many-valued logics, so that they can be used as LFI gently explosive logics.

We discovered how the possible world Kripke-style semantics for many-valued logics, based on a complete lattice of algebraic truth-values, can be interpreted as a paraconsistent approach to these logics. Finally we established the relationships between many-valuedness, autoreferential assumption and the paraconsistency.

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8. References

- Anderson A., Belnap N., *Entailment: the logic of relevance and necessity*, Princeton University Press, Princeton, NY, 1975.
- Beckert B., Hanhle R., Manyá F., “Transformations between signed and classical clause logic”, *Proceedings of the 29th IEEE International Symposium on Multiple-Valued Logics*,

- Freiburg, Germany, IEEE Computer Society, pp. 248–255, 1999.
- Belnap N., “A useful four-valued logic”, in J.-M. Dunn, G. Epstein (eds), *Modern Uses of Multiple-Valued Logic*, D. Reidel, 1977.
- Béziau J.-Y., “Are paraconsistent negations negations?”, in W. Carnielli (ed.), *Paraconsistency: the logical way to the inconsistent*, Marcel Dekker, New York, pp. 465–486, 2002.
- Birkhoff G., *Lattice theory*, 3rd edn, Amer Mathematical Society, 1940. reprinted 1979.
- Blair H., Subrahmanian V., “Paraconsistent logic programming”, *Theoretical Computer Science*, vol. 68, pp. 135–154, 1989.
- Brink C., Gabbay D., Olhbach H., Towards automating duality, Technical Report num. MPI-I-93-220, Max-Planck-Institut für Informatik, Saarbrücken, Germany, 1993.
- Bull R., Segerberg K., “Basic modal logic”, in V (ed.), *Handbook of Philosophical Logic*, vol. II, Reidel, Dordrecht, pp. 1–88, 1984.
- Cadoli M., Schaerf M., “On the complexity of entailment in propositional multivalued logics”, *Annals of Mathematics and Artificial Intelligence*, vol. 18, pp. 29–50, 1996.
- Carnielli W., Coniglio M., Marcos J., “Logics of Formal Inconsistency”, in D. Gabbay, F. Guenther (eds), *Handbook of Philosophical Logic*, vol. 14, Kluwer Academic Publishers, pp. 1–93, 2006.
- Carnielli W., Marcos J., “A Taxonomy of C-Systems”, in W. Carnielli, M. Coniglio, I. D’Ottaviano (eds), *Paraconsistency – the Logical Way to the Inconsistent*, vol. 228 of *Lecture Notes in Pure and Applied Mathematics*, Marcel Dekker, New York, 2002.
- Celani S., Jansana R., “A new semantics for Positive Modal Logic”, *Notre Dame Journal of Formal Logic*, 1997.
- Clark D., Davey B., *Natural dualities for the working algebraist*, vol. 57 of *Cambridge studies in advanced mathematics*, 1st edn, Cambridge University Press, 1998.
- da Costa N., “On the theory of inconsistent formal systems”, *Notre Dame Journal of Formal Logic*, vol. 15, pp. 497–510, 1974.
- Došen K., “Negation as a modal operator”, *Reports on Mathematical Logic*, vol. 20, pp. 15–28, 1986.
- Dugundji J., “Note on a property of matrices for Lewis and Langford’s calculi of propositions”, *Journal of Symbolic Logic*, vol. 5, pp. 150–151, 1940.
- Dunn J., “Gaggle theory: An abstraction of Galois connections and residuation with applications to negation and various logical operations”, in J. van Eijck (ed.), *Proceedings of Logics in AI, European Workshop, JELIA ’90, Amsterdam, The Netherlands, September 10-14, 1990*, vol. 478 of *Lecture Notes in Computer Science*, Springer, 1991.
- Dunn J., “Partial-Gaggles Applied to Logics with Restricted Structural Rules”, in P. Schroeder-Heister, K. Došen (eds), *Substructural Logics*, Oxford University Press, pp. 63–108, 1993.
- Dunn J., “Positive Modal Logic”, *Studia Logica*, vol. 55, pp. 301–317, 1995.
- Fitting M., “Bilattices and the semantics of logic programming”, *Journal of Logic Programming*, vol. 11, pp. 91–116, 1991.

- Gentzen G., “Über die Existenz unabhängiger Axiomensysteme zu unendlichen Satzsystemen”, *Mathematische Annalen*, vol. 107, pp. 329–350, 1932.
- Ginsberg M., “Multivalued logics: A uniform approach to reasoning in artificial intelligence”, *Computational Intelligence*, vol. 4, pp. 265–316, 1988.
- Goldblatt R., “Semantic analysis of Orthologic”, *Journal of Philosophical Logic*, vol. 3, num. 1–2, pp. 19–35, 1974.
- Goldblatt R., “Metamathematics of modal logic, Part II”, *Reports on Mathematical Logic*, 1976a.
- Goldblatt R., “Metamathematics of modal logic, Part II”, *Reports on Mathematical Logic*, 1976b.
- Haim M., “Duality for lattices with operators: a modal logic approach”, <http://citeseer.ist.psu.edu/haim00duality.html>, 2000.
- Hertz P., “Über Axiomensysteme für beliebige Satzsysteme”, *Mathematische Annalen*, vol. 101, pp. 457–514, 1929.
- Imaz G., Manyá F., “The satisfiability problem for multiple-valued Horn formulae”, *Proceedings of the 24th IEEE International Symposium on Multiple-Valued Logics (ISMVL)*, Boston, IEEE Computer Society, pp. 250–256, 1994.
- Jipsen J., Tsinakis C., “A survey of Residuated lattices”, in J. Martinez (ed.), *Ordered Algebraic Structures*, Kluwer Academic Publishers, pp. 19–56, 2002.
- Kalmbach G., *Orthomodular lattices*, Academic Press, London, 1983.
- Kifer M., Subrahmanian V., “Theory of generalized annotated logic programming and its applications”, *Journal of Logic Programming*, vol. 12, num. 4, pp. 335–368, 1992.
- Kripke S., “Semantical analysis of Intuitionistic logic”, in J. Crossley, M. Dummett (eds), *Formal Systems and Recursive Functions*, North Holland, pp. 92–130, 1965.
- Lewis C., “Implication and the algebra of logic”, *Mind (New Series)*, vol. 21, pp. 522–531, 1912.
- Łukasiewicz J., “A system of modal logic”, *Journal of Computing Systems*, vol. 1, pp. 111–149, 1953.
- Majkić Z., “Ontological encapsulation of many-valued logic”, 2004, 19th Italian Symposium of Computational Logic (CILC04), June 16–17, Parma, Italy.
- Majkić Z., “Autoepistemic logic programming for reasoning with inconsistency”, 2005a, International Symposium on Logic-based Program Synthesis and Transformation (LOPSTR), September 7–9, 2005, Imperial College, London, UK.
- Majkić Z., “Beyond Fuzzy: Parameterized approximations of Heyting algebras for uncertain knowledge”, 2005b, 2nd Indian International Conference on Artificial Intelligence (IICAI-05), December 20–22, Pune, India.
- Majkić Z., “Functional Many-valued Logic and Global Predicate Compression”, 2006a, Workshop on Multi-valued Logic Programming and Applications (MVLPA 2006, together with ICLP 2006), 17–20 August, Seattle, WA.

- Majkić Z., “Intuitionistic Truth-knowledge Symmetric Bilattices for Uncertainty in Intelligent Systems”, *Proceedings of the 3rd International IEEE Conference on Intelligent Systems (IS06)*, 4-6 Sept., London, UK, IEEE Computer Society, 2006b.
- Majkić Z., “Many-valued Logic Programming and Fixpoint Semantics for Higher-order Herbrand Models”, in M. Fink, H. Tompits, S. Woltran (eds), *Proceeding of the 20th Workshop on Logic Programming (WLP 2006)*, February 22-24, Vienna, Austria, vol. 1843-06-02, Technische Universität Wien, 2006c.
- Majkić Z., “New Representation Theorem for Many-valued Modal Logics”, 2006d, http://www.geocities.com/zoran_it.
- Majkić Z., “Reduction of Many-valued Logic Programs into 2-valued Modal Logics”, 2006e, <http://www.cs.umd.edu/~zoran>.
- Majkić Z., “Bilattices, Intuitionism and Truth-knowledge Duality: Concepts and Foundations”, *Journal of Multiple-valued Logic and Soft Computing*, vol. ??, pp. ??–??, 2007.
- Majkić Z., “Weakening of Intuitionistic Negation for Many-valued Paraconsistent Da Costa System”, *Notre Dame Journal of Formal Logics*, 2008.
- McKinsey J., Tarski A., “Some Theorems About the Sentential Calculi of Lewis and Langford”, *Journal of Symbolic Logic*, vol. 13, pp. 1–15, 1948.
- Nelson D., *Negation and separation of concepts in constructive systems*, Studies in Logic and the Foundations of Mathematics, Elsevier, Amsterdam, North Holland, 1959.
- Sofronie-Stokkermans V., “Representation Theorems and the semantics of non-classical logics, and applications to automated theorem proving”, in M. Fitting, E. Orłowska, J. Kacprzyk (eds), *Beyond two: theory and applications of multiple-valued logic*, Physica-Verlag, pp. 59–100, 2003.
- Urquhart A., “A topological representation theory for lattices”, *Algebra Universalis*, vol. 8, pp. 45–58, 1978.